Long time behaviour of Ricci flow on open 3-manifolds

Laurent Bessières, Gérard Besson and Sylvain Maillot

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Abstract

We study the long time behaviour of Ricci flow with bubbling-off on a possibly noncompact 3-manifold of finite volume whose universal cover has bounded geometry. As an application, we give a Ricci flow proof of Thurston's hyperbolisation theorem for 3-manifolds with toral boundary that generalizes Perelman's proof of the hyperbolisation conjecture in the closed case.

1 Introduction

A riemannian metric is *hyperbolic* if it is complete and has constant sectional curvature equal to -1. If N is a 3-manifold-with-boundary, then we say it is *hyperbolic* if its interior admits a hyperbolic metric. In the mid-1970s, W. Thurston stated his *Hyperbolisation Conjecture,* which gives a natural sufficient condition on the topology of a 3-manifold-with-boundary N which implies that it is hyperbolic. Recall that N is *irreducible* if every embedded 2-sphere in N bounds a 3-ball. A version of Thurston's conjecture states that if N is compact, connected, orientable, irreducible, and $\pi_1 N$ is infinite and does not have any subgroup isomorphic to $\mathbf{Z}^2$, then N is hyperbolic. If one replaces the hypotheses on the fundamental group by the assumption that N is *atoroidal*, i.e. every incompressible embedded 2-torus in N is parallel to a component of ∂N, then $\operatorname{int} N$ is hyperbolic or Seifert fibred.

Thurston proved his conjecture for the case of so-called *Haken manifolds*, which includes the case where ∂N is nonempty. The case where N is closed was solved by G. Perelman using Ricci flow with surgery, based on ideas of R. Hamilton.

It is natural to ask whether the Hamilton-Perelman proof works when $\partial N \neq \emptyset$. The interior M of N, on which one wishes to construct a hyper-

bolic metric, is then noncompact. This question can be divided into two parts: first, is it possible to construct some version of Ricci flow with surgery on such an open manifold M, under reasonable assumptions on the initial metric? Second, does it converge (modulo scaling) to a hyperbolic metric? A positive answer to both questions would give a Ricci flow proof of the full Hyperbolisation Conjecture logically independent of Thurston's results.

A positive answer to the first question was given in [BBM11], for initial metrics of bounded geometry, i.e. of bounded curvature and positive injectivity radius. If one considers irreducible manifolds, surgeries are topologically trivial: each surgery sphere bounds a 3-ball. Hence a surgery splits off a 3-sphere. In this situation we can refine the construction of the Ricci flow with surgery so that it is not necessary to perform the surgery topologically. We obtain a solution which is a piecewise smooth Ricci flow on a fixed manifold; at singular times, one performs only a metric surgery, changing the metric on the 3-balls. This construction was defined in [BBB+10] in the case of closed irreducible non spherical 3-manifolds, and called Ricci flow with bubbling-off. One can extend it to the setting of bounded geometry. The purpose of this paper is to answer the second question, in the situation where the initial metric has a *cusp-like structure*.

Definition 1.1. We say that a metric g on M has a *cusp-like structure*, or is a *cusp-like metric*, if M has finitely many ends (possibly zero), and each end has a neighbourhood which admits a metric g_{cusp} homothetic to a rank two cusp neighbourhood of a hyperbolic manifold such that $g - g_{\text{cusp}}$ goes to zero at infinity in C^k-norm for all integer k. (Thus if M is closed, any metric is cusp-like.)

Note that such a metric is automatically complete with bounded curvature and of finite volume, but its injectivity radius equals zero hence does not have bounded geometry. However its universal covering does. Also note that if M admits a cusp-like metric, then M admits a manifold compactification whose boundary is empty or a union of 2-tori. This compactification is irreducible (resp. atoroidal, resp. Seifert-fibred) if and only if M is irreducible (resp. atoroidal, resp. Seifert-fibred).

In section 2 we construct a Ricci flow with bubbling-off on M, for any initial cusp-like metric, passing to the universal cover and working equivariantly. For simplicity we restrict ourselves to the case where M is non spherical. We also prove that the cusp-like structure is preserved by this flow (cf. Theorem 2.19).

Using this tool, we can adapt Perelman's proof of geometrisation to obtain the following result:

Theorem 1.2. *Let M be a connected, orientable, irreducible, atoroidal* 3*-manifold and g_0 be a metric on M which is cusp-like at infinity. Then M is Seifert-fibred, or there exists a Ricci flow with bubbling-off $g(\cdot)$ on M defined on $[0,\infty)$, such that $g(0) = g_0$, and as t goes to infinity, $t^{-1}g(t)$ converges smoothly in the pointed topology to some finite volume hyperbolic metric on M. Moreover $g(\cdot)$ has finitely many surgeries and there are constants $T, C < \infty$ such that $[\mathrm{Rm}| < Ct^{-1}$ for all $t \geq T$.*

If N is a compact, connected, orientable 3-manifold such that ∂N is empty or a union of 2-tori, then $M = \operatorname{int} N$ always carries a cusp-like at infinity metric. Thus we obtain:

Corollary 1.3 (Thurston, Perelman)**.** *Let N be a compact, connected, orientable* 3*-manifold-with-boundary such that ∂N is empty or a union of* 2*-tori. If N is irreducible and atoroidal, then N is Seifert-fibred or hyperbolic.*

Note that it should be possible to obtain this corollary directly from the closed case by a doubling trick. The point of this paper is to study the behaviour of Ricci flow in the noncompact case.

Let us review some results concerning global stability or convergence to finite volume hyperbolics metrics. In the case of surfaces, R. Ji, L. Mazzeo and N. Sesum [JMS09] show that if (M, g_0) is complete asymptotically hyperbolic of finite aera with $\chi(M) < 0$ then the normalised Ricci flow with initial condition g_0 converges exponentially to the unique complete hyperbolic metric in its conformal class. G. Giesen and P. Topping [GT11] theorem 1.3 show that if g_0, possibly non complete and with unbounded curvature, is in the conformal class of a complete finite aera hyperbolic metric g_{hyp}, then there exists a unique Ricci flow with initial condition g_0 which is instantaneously complete and maximaly stretched, defined on $[0, +\infty)$ and such that the rescaled solution $(2t)^{-1}g(t)$ converges smoothly locally to g_0 as $t \to \infty$. Moreover if $g_0 \leq C g_{\mathrm{hyp}}$ for some constant $C > 0$ then the converence is global: For any $k \in \mathbf{N}$ and $\mu \in (0,1)$ there exists a constant $C > 0$ such that for all $t \geq 1$, $|(2t)^{-1}g(t) - g_{\mathrm{hyp}}|_{C^k(M,g_{\mathrm{hyp}})} < \frac{C}{t^{1-\mu}}$. In dimension ≥ 3, R. Bamler [Bam11b] show that if g_0 is a small C^0-perturbation of a complete finite volume hyperbolic metric g_{hyp}, that is if $|g_0 - g_{\mathrm{hyp}}|_{C^0(M,g_{\mathrm{hyp}})} < \varepsilon$ where $\varepsilon = \varepsilon(M, g_{hyp}) > 0$, then the normalised Ricci flow with initial condition g_0 is defined for all time and converges in the pointed Gromov-Hausdorff topology to g_{hyp}. In dimension 3 at least, there cannot be any global convergence result. Indeed, consider a complete finite volume hyperbolic manifold (M^3, g_{hyp}) with one cusp at least. Let g_0 be a small C^0 pertubation of g_{hyp} such that g_0 remains cusp-like at infinity but with a different hyperbolic structure in the given cusp

(change the cross-sectional flat structure on the cusp). By Bamler [Bam11b] a rescaling of $g(t)$ converges in the pointed topology to g_{hyp}. However our stability theorem 2.19 implies that the cusp-like structure of g_0 is preserved for all time, hence convergence cannot be global.

The paper is organised as follows. In Section 2 we introduce the necessary definitions and we prove the existence of a Ricci flow with bubbling-off which preserves cusp-like structures. Section 3 is devoted to a thick-thin decomposition theorem which shows that the thick part of $(M, t^{-1}g(t))$ (sub)-converges to a complete finite volume hyperbolic manifold. We give also some estimates on the longtime behaviour of our solutions. In Section 4 we prove the incompressibility of the tori bordering the thick part. Section 5 is consacred to collapsing theory, which is used to obtain that the thin part is graphed. There is also a topological description of the thin part used in the next section to ruled out surgeries there. Finally the main theorem 1.2 is proved in Section 6. An overview of the proof is given at the beginning of this section.

We end this introduction by the following convention: *all* 3*-manifolds in this article are connected and orientable.*

2 Ricci flow with bubbling-off on open manifolds

2.1 Definition and existence

In this section we define Ricci flow with bubbling-off and state the main existence theorem.

For convenience of the reader we recall here the most important definitions involved, and refer to Chapters 2, 4, and 5 of the monograph [BBB+10] for completeness.

Definition 2.1 (Evolving metric)**.** Let M be an n-manifold and $I \subset \mathbf{R}$ be an interval. An *evolving metric* on M defined on I is a map $t \mapsto g(t)$ from I to the space of smooth riemannian metrics on M. A *regular* time is a value of t such that this map is $\mathcal{C}^1$-smooth in a neighbourhood of t. If t is not regular, then it is *singular*. We denote by $g_+(t)$ the right limit of g at t, when it exists. An evolving metric is *piecewise* C^1 if singular times form a discrete subset of $\mathbf{R}$ and if $t \mapsto g(t)$ is left continuous and has a right limit at each point. A subset $N \times J \subset M \times I$ is *unscathed* if $t \to g(t)$ is smooth there. Otherwise it is *scathed*.

If g is a riemannian metric, we denote by $R_{\min}(g)$ (resp. $R_{\max}(g)$) the infimum (resp. the supremum) of the scalar curvature of g.

Definition 2.2 (Ricci flow with bubbling-off)**.** A piecewise $\mathcal{C}^1$ evolving metric $t \mapsto g(t)$ on M defined on I is a *Ricci flow with bubbling-off* if

(i) The Ricci flow equation $\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}$ is satisfied at all regular times;

(ii) for every singular time $t \in I$ we have

  (a) $R_{\min}(g_+(t)) \geqslant R_{\min}(g(t))$, and

  (b) $g_+(t) \leqslant g(t)$.

Remark 2.3. If $g(\cdot)$ is a complete Ricci flow with bubbling-off of bounded sectional curvature defined on an interval of the type $[0,T]$ or $[0,\infty)$, and if $g(0)$ has finite volume, then $g(t)$ has finite volume for every t.

A *parabolic neighbourhood* of a point $(x,t) \in M \times I$ is a set of the form

$$P(x,t,r,-\Delta t) = \{(x',t') \in M \times I \mid x' \in B(x,t,r), t' \in [t-\Delta t, t]\}.$$

Definition 2.4 (κ-noncollapsing)**.** For $\kappa, r > 0$ we say that $g(\cdot)$ is *κ-collapsed* at (x,t) on the scale r if for all (x',t') in the parabolic neighbourhood $P(x,t,r,-r^2)$ we have $|\mathrm{Rm}(x',t')| \leq r^{-2}$ and $\mathrm{vol}(B(x,t,r)) < \kappa r^n$. Otherwise, $g(\cdot)$ is *κ-noncollapsed* at (x,t) on the scale r. If this is true for all $(x,t) \in M \times I$, then we say that $g(\cdot)$ is *κ-noncollapsed on the scale r*.

Next is the definition of *canonical neighbourhoods*. From now on and until the end of this section, M is a 3-manifold and ε, C are positive numbers.

Definition 2.5 (ε-necks, ε-caps)**.** Let g be a riemannian metric on M. If x is a point of M, then an open subset $U \subset M$ is an *ε-neck centred at x* if (U,g,x) is ε-homothetic to $(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}), g_{\mathrm{cyl}}, (*,0))$, where g_{cyl} is the standard metric with unit scalar curvature. This open set U is an *ε-cap centred at x* if U is the union of two sets V, W such that $x \in \mathrm{int}\, V$, V is a closed 3-ball, $\bar{W} \cap V = \partial V$, and W is an ε-neck.

Definition 2.6 ((ε, C)-cap)**.** An open subset $U \subset M$ is an *(ε, C)-cap centred at x* if U is an ε-cap centred at x which satisfies the following estimates: $R(x) > 0$ and there exists $r \in (C^{-1}R(x)^{-1/2}, CR(x)^{-1/2})$ such that

(i) $\overline{B(x,r)} \subset U \subset B(x,2r)$;

(ii) The scalar curvature function restricted to U has values in a compact subinterval of $(C^{-1}R(x), CR(x))$;

(iii) $\mathrm{vol}(U) > C^{-1}R(x)^{-3/2}$ and if $B(y,s) \subset U$ satisfies $|\mathrm{Rm}| \leqslant s^{-2}$ on $B(y,s)$ then

$$C^{-1} < \frac{\mathrm{vol}B(y,s)}{s^3} \ ; \tag{1}$$

(iv) On U,

$$|\nabla R| < CR^{\frac{3}{2}} \ , \tag{2}$$

(v) On U,

$$|\Delta R + 2|\,\mathrm{Ric}\,|^2| < CR^2 \ , \tag{3}$$

(vi) On U,

$$|\nabla \mathrm{Rm}\,| < C|\,\mathrm{Rm}\,|^{\frac{3}{2}} \ , \tag{4}$$

Remark 2.7. If $t \mapsto g(t)$ is a Ricci flow, then $\frac{\partial R}{\partial t} = \Delta R + 2|\,\mathrm{Ric}\,|^2$ hence equation (3) implies that $|\dfrac{\partial R}{\partial t}| \leq CR^2$.

Definition 2.8 (Strong ε-neck)**.** We call *cylindrical flow* the pointed evolving manifold $(S^2 \times \mathbf{R}, \{g_{\mathrm{cyl}}(t)\}_{t\in(-\infty,0]})$, where $g_{\mathrm{cyl}}(\cdot)$ is the product Ricci flow with round first factor, normalised so that the scalar curvature at time 0 is 1. If $g(\cdot)$ is an evolving metric on M, and (x_0, t_0) is a point in spacetime, then an open subset $N \subset M$ is a *strong ε-neck centred at* (x_0, t_0) if there exists $Q > 0$ such that $(N, \{g(t)\}_{t\in[t_0-Q^{-1},t_0]}, x_0)$ is unscathed, and, denoting $\bar{g}(t) = Qg(t_0 + tQ^{-1})$ the parabolic rescaling with factor $Q > 0$ at time t_0, $(N, \{\bar{g}(t)\}_{t\in[-1,0]}, x_0)$ is ε-close to $(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}), \{g_{\mathrm{cyl}}(t)\}_{t\in[-1,0]}, *)$.

Definition 2.9 ((ε, C)-canonical neighbourhood)**.** Let $\{g(t)\}_{t\in I})$ be an evolving metric on M. We say that a point (x,t) admits (or is centre of) an (ε, C)*-canonical neighbourhood* if x is centre of an (ε, C)-cap in $(M, g(t))$ or if (x,t) is centre of a strong ε-neck which satisfies (i)–(vi).

In [BBB+10, Section 5.1] we fix constants ε_0, C_0. For technical reasons, we need to take them slightly different here; this will be explained in the proof of Theorem 2.15.

Definition 2.10 (Canonical Neighbourhood Property $(CN)_r$)**.** Let $r > 0$. An evolving metric satisfies the property $(CN)_r$ if, for any (x,t), if $R(x,t) \geq r^{-2}$ then (x,t) is centre of an (ε_0, C_0)-canonical neighbourhood.

Next we define a pinching property for the curvature tensor coming from work of Hamilton and Ivey. We consider a familly of positive functions $(\phi_t)_{t\geqslant 0}$ defined as follows. Set $s_t := \frac{e^2}{1+t}$ and define

$$\phi_t : [-2s_t, +\infty) \longrightarrow [s_t, +\infty)$$

as the reciprocal of the increasing function

$$s \mapsto 2s(\ln(s) + \ln(1+t) - 3).$$

A key property of this function is that $\frac{\phi_t(s)}{s} \to 0$ as $s \to +\infty$.

Definition 2.11 (Curvature pinched toward positive)**.** Let $I \subset [0, \infty)$ be an interval and $\{g(t)\}_{t \in I}$ be an evolving metric on M. We say that $g(\cdot)$ has *curvature pinched toward positive at time* t if for all $x \in M$ we have

$$R(x,t) \geqslant -\frac{6}{4t+1}, \tag{5}$$

$$\mathrm{Rm}(x,t) \geqslant -\phi_t(R(x,t)). \tag{6}$$

We say that $g(\cdot)$ has *curvature pinched toward positive* if it has curvature pinched toward positive at each $t \in I$.

This allows in particular to define the notion of *surgery parameters* r, δ (cf. [BBB$^+$10, Definition 5.2.5]). Using [BBB$^+$10, Theorem 5.2.4] we also define their *associated cutoff parameters* h, Θ. Using the metric surgery theorem, we define the concept of a metric g_+ being *obtained from* $g(\cdot)$ *by* (r, δ)*-surgery at time* t_0 (cf. [BBB$^+$10, Definition 5.2.7]). This permits to define the following central notion:

Definition 2.12 (Ricci flow with (r, δ)-bubbling-off)**.** Fix surgery parameters r, δ and let h, Θ be the associated cutoff parameters. Let $I \subset [0, \infty)$ be an interval and $\{g(t)\}_{t \in I}$ be a Ricci flow with bubbling-off on M. We say that $\{g(t)\}_{t \in I}$ is a *Ricci flow with* (r, δ)*-bubbling-off* if it has the following properties:

i) $g(\cdot)$ has curvature pinched toward positive and satisfies $R(x,t) \leqslant \Theta$ for all $(x,t) \in M \times I$;

ii) For every singular time $t_0 \in I$, the metric $g_+(t_0)$ is obtained from $g(\cdot)$ by (r, δ)-surgery at time t_0;

iii) $g(\cdot)$ satisfies property $(CN)_r$.

Definition 2.13 (Ricci flow with (r, δ, κ)-bubbling-off)**.** Let $\kappa > 0$. A Ricci flow with (r, δ)-bubbling-off $g(\cdot)$ is called a *Ricci flow with* (r, δ, κ)*-bubbling-off* if it is κ-noncollapsed on all scales less than or equal to 1.

Definition 2.14. A metric g on a 3-manifold M is *normalised* if it satisfies $\operatorname{tr} \mathrm{Rm}^2 \leq 1$ and each ball of radius 1 has volume at least half of the volume of the unit ball in Euclidean 3-space.

Note that a normalised metric always has bounded geometry.

At last we can state our existence theorem:

Theorem 2.15. *There exist decreasing sequences of positive numbers $r_k, \kappa_k > 0$ and, for every continuous positive function $t \mapsto \bar{\delta}(t)$, a decreasing sequence of positive numbers δ_k with $\delta_k \leqslant \bar{\delta}(\cdot)$ on $]k, k+1]$ with the following property. For any complete, normalised, nonspherical, irreducible riemannian 3-manifold (M, g_0), one of the following conclusions holds:*

i. There exists $T > 0$ and a complete Ricci flow with bubbling-off $g(\cdot)$ of bounded geometry on M, defined on $[0, T]$, with $g(0) = g_0$, and such that every point of $(M, g(T))$ is centre of an ε_0-neck or an ε_0-cap, or

ii. There exists a complete Ricci flow with bubbling-off $g(\cdot)$ of bounded geometry on M, defined on $[0, +\infty)$, with $g(0) = g_0$, and such that for every nonnegative integer k, the restriction of $g(\cdot)$ to $]k, k+1]$ is a Ricci flow with $(r_k, \delta_k, \kappa_k)$-bubbling-off.

Definition 2.16 (Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off)**.** We fix forever a function $r(\cdot)$ such that $r(t) = r_k$ on all intervals $]k, k+1]$. Given $\delta(\cdot)$ satisfying $\delta(t) = \delta_k$ on all $]k, k+1]$, we call a solution as above a *Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off.*

Addendum 2.17 (Ricci flow with bubbling-off on the quotient)**.** *With the same notation as in Theorem 2.15 and under the same hypotheses, if in addition (M, g_0) is a riemannian cover of some riemannian manifold (X, h_0), then in either case there exists a Ricci flow with bubbling-off $h(\cdot)$ on X such that for each t, $(M, g(t))$ is a riemannian cover of $(X, h(t))$, and in Case (ii), the restriction of $h(\cdot)$ to $]k, k+1]$ is a Ricci flow with (r_k, δ_k)-bubbling-off for every k.*

The only differences between Theorem 2.15 and Theorem 11.5 of [BBM11] is that M is assumed to be irreducible, that 'surgical solution' is replaced with 'Ricci flow with bubbling-off', and that there is the alternative conclusion (i).

Theorem 2.15 follows from iteration of the following result, which is analogous to [BBM11, Theorem 5.6]:

Theorem 2.18. *For every Q_0, ρ_0 and all $0 \leq T_A < T_\Omega < +\infty$, there exist $r, \kappa > 0$ and for all $\bar{\delta} > 0$ there exists $\delta \in (0, \bar{\delta})$ with the following property. For any complete, nonspherical, irreducible riemannian 3-manifold (M, g_0) which satisfies $|\operatorname{Rm}| \leq Q_0$, has injectivity radius at least ρ_0, has curvature pinched toward positive at time T_A, one of the following conclusions holds:*

i. There exists $T \in (T_A, T_\Omega)$ and a Ricci flow with bubbling-off $g(\cdot)$ on M, defined on $[T_A, T]$, with $g(T_A) = g_0$, and such that every point of $(M, g(T))$ is centre of an ε_0-neck or an ε_0-cap, or

ii. There exists a Ricci flow with (r, δ, κ)-bubbling-off $g(\cdot)$ on M, defined on $[T_A, T_\Omega]$, satisfying $g(T_A) = g_0$.

Theorem 2.18 follows from three propositions A, B, C, which are analogous to Propositions A, B, C of [BBM11]. The only notable difference is that in Proposition A, we add the alternative conclusion that in $(M, g(b))$, every point is centre of an ε_0-cap or an ε_0-neck. Let us exlain the proof of this adapted proposition A (compare with [BBM11]). It uses the surgical procedure of the monograph [BBB$^+$10] rather than that of [BBM11]. If the curvature is large everywhere, then every point has a canonical neighbourhood, so the alternative conclusion holds. Otherwise, we find a locally finite collection of cutoff δ-necks $\{N_i\}$ which separates the part with high curvature from the part with low curvature. Since M is irreducible and not homeomorphic to S^3, the middle sphere of each N_i bounds a unique topological 3-ball B_i. Then one of the following cases occurs:

Case 1 Each B_i is contained in a unique maximal 3-ball B_j.

In this case the surgical procedure using the Metric surgery theorem of [BBB$^+$10] is performed on each maximal cap B_j, yielding a metric which has the desired properties.

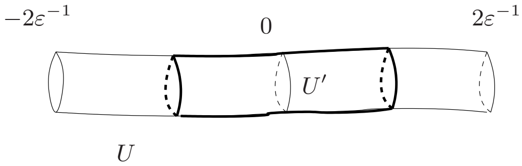
Case 2 M is the union of the B_i's.

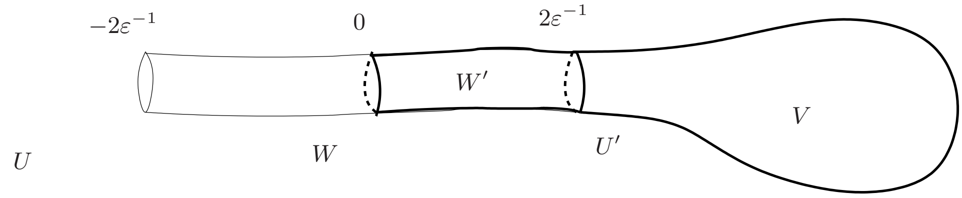
Then each point is separated from infinity by a cutoff neck, so each point is centre of a cap. Hence the alternative conclusion holds.

Finally, we need to explain how the addendum is proved. We already remarked in [BBM11] Section 11 that the construction can be made equivariant with respect to a properly discontinuous group action, by work of Dinkelbach and Leeb [DL09]. The only thing to check is that we still have the Canonical Neighbourhood Property for the quotient evolving metric $h(\cdot)$. This is not obvious, since the projection map $p : M \to X$ might not be injective when restricted to a canonical neighbourhood.

We use a classical trick: by adjusting the constants, we may assume that $g(\cdot)$ has the stronger property that each point (x, t) such that $R(x, t) \geq r^{-2}$ has an $(\varepsilon_0/2, C_0)$-canonical neighbourhood. Take now $(x, t) \in X \times I$ such that $R(x, t) \geq r^{-2}$. Choose $\bar{x} \in M$ such that $p(\bar{x}) = x$. Then $R(\bar{x}, t) = R(x, t) \geq r^{-2}$, so $(\bar{x}, t)$ has an $(\varepsilon_0/2, C_0)$-canonical neighbourhood U. By

truncation, it also has an (ε_0, C_0)-canonical neighbourhood U' contained in U (see figure below) :





Precisely, if U is an $\varepsilon_0/2$-neck with parametrisation $\phi : S^2 \times (-2\varepsilon_0^{-1}, 2\varepsilon_0^{-1}) \to U$, we set $U' := \phi(S^2 \times (-\varepsilon_0^{-1}, \varepsilon_0^{-1}))$. If U is a cap, then U is the union of two sets V, W, where $\bar{W} \cap V = \partial V$ and W is an $\varepsilon_0/2$-neck with parametrisation ϕ. Then we set $W' = \phi(S^2 \times (0, 2\varepsilon_0^{-1}))$ and $U' = V \cup W'$.

Claim 1. The restriction of the projection map p to U' is injective.

Once the claim is proved, we can just project U' to X and obtain an (ε_0, C_0)-canonical neighbourhood for (x, t), so we are done.

To prove the claim we consider two cases:

Case 1 U and U' are caps.

Assume by contradiction that there is an element γ in the deck transformation group, different from the identity, and a point $y \in U'$ such that $\gamma y \in U'$. Following [DL09], we consider the subset N_{ε_0} of M consisting of points which are centres of ε_0-necks. This set has an equivariant foliation $\mathcal{F}$ by almost round 2-spheres. All points sufficiently close to the centre of W are centres of ε_0-necks.

Pick a point z in $N_{\varepsilon_0} \cap W \setminus W'$ sufficiently far from W' so that the leaf S of $\mathcal{F}$ through z is disjoint from U'. By Alexander's theorem, S bounds a 3-ball $B \subset U$. Note that B contains U'. If $S = \gamma S$, then $B = \gamma B$ or $M = B \cup \gamma B$. The former possibility is ruled out by the fact that the action is free, while any self-homeomorphism of the 3-ball has a fixed point. The latter is ruled out by the assumption that M is not diffeomorphic to S^3.

Hence $S \neq \gamma S$. Since S and γS are leafs of a foliation, they are disjoint. Then we have the following three possibilities:

Subcase a γS is contained in B.

Then we claim that $\gamma B \subset B$. Indeed, otherwise we would have $M = B \cup \gamma B$, and M would be diffeomorphic to S^3. Now γ acts by isometry, so $\mathrm{vol} B = \mathrm{vol}\gamma B$. This is impossible since the annular region between S and γS has nonzero volume.

Subcase b S is contained in γB. This case is ruled out by a similar argument exchanging the roles of S and γS (resp. of B and γB.)

Subcase c B and γB are disjoint.

Then since $U' \subset B$, the sets U' and $\gamma U'$ are also disjoint, contradicting the existence of y.

Case 2 U and U' are necks. Seeking a contradiction, let γ be an element of the deck transformation group, different from the identity, and y be a point of U' such that $\gamma y \in U'$. Consider again the set N_{ε_0} defined above and its equivariant foliation $\mathcal{F}$. Since U' is contained in the bigger set U, each point of U' is centre of an ε_0-neck. Let S (resp. S') be the leaf of $\mathcal{F}$ passing through y (resp. γy.) Since M is irreducible, S (resp. S') bounds a 3-ball B (resp B'). As in the previous case, we argue that one of these balls is contained into the other, otherwise we could cover M by B, B' and possibly an annular region between them, and get that M is diffeomorphic to S^3. Since γ acts by an isometry, we must in fact have $B = B'$, and γ has a fixed point, contradicting our hypotheses. This finishes the proof of the claim, hence that of Addendum 2.17.

2.2 Stability of cusp-like structures

In this section, we prove the stability of cusp-like structures under Ricci flow with bubbling-off. We consider a (nonspherical, irreducible) 3-manifold M, endowed with a cusp-like metric g_0. Let us denote g_c a metric on M which is hyperbolic on the complement of some compact of M, and such that, for each end of M there is a factor $\lambda > 0$ such that $\lambda g_0 - g_c$ goes to zero at infinity in the end, in C^k-norm for each integer k. Let $g(\cdot)$ be a Ricci flow with $r(\cdot), \delta(\cdot)$-bubbling-off such that $g(0) = g_0$, defined on $[0, T]$ for some $T > 0$.

Under these assumptions, we have:

Theorem 2.19 (Stability of cusp-like structures)**.** *For each end of M, for every $t \in [0, T]$, there is a factor $\lambda > 0$ such that $\lambda g(t) - g_c$ goes to zero at infinity in this end, in C^k-norm for each integer k.*

Proof of theorem 2.19. Let us first explain the idea. It is enough to work on each cusp. The main tool is the persistence theorem 8.1.3 from [BBB$^+$10], which proves that a Ricci flow remains close, on a parabolic neighbourhood where it has a priori curvature bounds, to a given Ricci flow model, if the initial data are sufficiently close on some larger balls. The model we use now is a hyperbolic Ricci flow on $T^2 \times \mathbf{R}$. To obtain the required curvature bounds, we shall consider an interval $[0, t]$ where the closeness to the hyperbolic flow holds, and $\sigma > 0$ fixed small enough so that property $(CN)_r$, which prevents scalar curvature to explode too fast, give curvature bounds on $[0, t + \sigma]$. The Persistence theorem then gives closeness to the hyperbolic flow until time $t + \sigma$ on a smaller neighbourhood of the cusp. One can iterate this procedure, shrinking the neighbourhood of the cusp of a definite amount at each step, until time T.

Remark 2.20. In fact we get a slightly more precise result: given any $A > 0$ there is a compact $K \subset M$ such that on each component of $M \setminus K$, $g(\cdot)$ is A^{-1}-homothetic to a hyperbolic Ricci flow on $\mathbf{T}^2 \times [a, +\infty) \times [0, T]$, for some $a > 0$.

Now we give details. Consider a neighbourhood U of a end of M, where g_c is hyperbolic. We can assume that (U, g_c) is isometric to $(\mathbf{T}^2 \times [0, +\infty), g_{\text{hyp}} = e^{-r} g_{\mathbf{T}^2} + dr^2)$, where $g_{\mathbf{T}^2}$ is flat. We denote $\mathcal{C}_s = \mathbf{T}^2 \times [s, +\infty)$, for $s \geq 0$. The hypothesis on g_0 is then equivalent to the requirement that for any $A > 0$, the end has a neighbourhood A^{-1}-homothetic to $(\mathcal{C}_s, g_{\text{hyp}})$ for some $s > 0$. Hence there is an embedding $\phi : \mathcal{C}_0 \to M$ parametrising the cusp and a factor $\lambda > 0$ such that, for all $A > 0$, $\lambda \phi^* g_0$ is A^{-1}-close to g_{hyp} on $\mathcal{C}_s$ for s large enough. For simplicity, we assume the scaling factor to be 1, and we define $\bar{g}(t) = \phi^* g(t)$ the pull-back metrics on $\mathcal{C}_0$.

We use as Ricci flow model, in the sense of [BBB$^+$10] (Theorem 8.1.3), the Ricci flow, denoted by $\bar{g}_0(\cdot)$, on $\mathbf{T}^2 \times \mathbf{R}$ whose initial metric is $\bar{g}_0(0) = g_{\text{hyp}} = e^{-s} g_{\mathbf{T}^2} + ds^2$. Hence we want to compare $\bar{g}(t)$ and $g_0(t)$ on $\mathcal{C}_s \times [0, T]$.

By definition of our Ricci flow with bubbling-off, $r(\cdot)$ and $\Theta(\cdot)$ are piecewise constant. More precisely, there exist $0 = t_0 < t_1 < \cdots < t_N = T$ such that $r(t) = r_i$ and $\Theta(t) = \Theta_i$ on $(t_i, t_{i+1}]$. In particular, $g(t)$ satisfies the canonical neighbourhood property at scale r_i on this interval (every point at which the scalar curvature is greater than r_i^{-2} is centre of an (ε_0, C_0) canonical neighbourhood) and the scalar curvature is bounded above by Θ_i. The pinching assumption (cf. Definition 2.11) then implies that the full curvature tensor is bounded by some K_i on the same interval.

Set $K := \sup_{i=1,\dots,N-1}\{K_i\}$. Define a small number $\sigma > 0$ by setting

$$\sigma := \frac{r_{N-1}^2}{2C_0} \leq \frac{r_i^2}{2C_0} \quad \forall i = 0, \dots, N-1\,.$$

This number is small enough such that $g(\cdot)$ cannot develop a singularity on a cusp on $[t, t+\sigma]$ if $R \leq 0$ at time t. Precisely:

Lemma 2.21. *Let $s \geq 0$. If $\bar{g}(\cdot)$ is unscathed on $\mathcal{C}_s \times [0, \Delta]$ and has scalar curvature $R \leqslant 0$ there, then it is also unscathed on $\mathcal{C}_s \times [0, \Delta + \sigma]$ and has curvature tensor bounded by K.*

Proof of Lemma 2.21. We know that singular times are discrete. Let $t \in [0, \sigma]$ be maximal such that $\mathcal{C}_s \times [0, \Delta + t]$ is unscathed for $\bar{g}(\cdot)$ (possibly $t = 0$).

Assertion 2.22. *For $t' \in [\Delta, \Delta + t]$ and $x \in \mathcal{C}_s$ we have*

$$R(x, t') \leq 2r(t')^{-2} << h(t')^{-2}\,.$$

Proof of the assertion. Since $r(\cdot)$ is nonincreasing, $g(\cdot)$ satisfies $(CN)_{r(t)}$ on $[\Delta, t']$. If $R(x, \Delta) \leq 0$ and $R(x, t') > 2r(t')^{-2}$ we can find a subinterval $[t_1, t_2] \subset [\Delta, t']$ such that for $s \in [t_1, t_2]$, $R(x, s) \geq r(t')^{-2}$ and

$$r(x, t_1) = r(t')^{-2} \quad \text{and} \quad r(x, t_2) = 2r(t')^{-2}\,.$$

Then the inequality $|\frac{\partial R}{\partial t}| < C_0 R^2$ holds on $\{x\} \times [t_1, t_2]$, thanks to property (3) of canonical neighbourhoods (cf Remark 2.7). The contradiction follows by integrating this inequality and using $t_2 - t_1 < \sigma$. □

Assume that $t < \sigma$, hence there is surgery at time $\Delta + t$. the surgery spheres are disjoints from $\phi(\mathcal{C}_s)$, as they have curvature $\approx h(\Delta + t)^{-2}$ and curvature on $\phi(\mathcal{C}_s)$ is $<< h(\Delta + t)^{-2}$. By definition of our surgery, this means that $\phi(\mathcal{C}_s) \subset M$ is contained in a 3-ball where the metric surgery is performed. But a cusp of M cannot be contained in a 3-ball of M, hence we get a contradiction. We conclude that $t = \sigma$ and $R(x, t') \leq 2r(t')^{-2}$, $\forall t' \in [\Delta, \Delta + \sigma]$. The pinching assumption then implies $|\operatorname{Rm}| < K$ there. □

We now define $\rho = \rho(A, T, K)$ given by the persistence theorem 8.1.3 in [BBB+10]. The proof of Theorem 2.19 is obtained by iteration of Lemma 2.21 and the persistence theorem as follows.

Let $s_0 > 0$ large enough so that $\bar{g}(0)$ is ρ^{-1}-close to $g_0(0)$ on $\mathcal{C}_{s_0}$. In particular $R \leq 0$ there, so by Lemma 2.21, $\bar{g}(\cdot)$ is unscathed on $\mathcal{C}_{s_0} \times [0, \sigma]$, with curvature tensor bounded by K. The above-mentioned persistence theorem

applied to $P(q, 0, A, \sigma)$, for all $q \in \mathcal{C}_{s_0+\rho}$, shows that $\bar{g}(t)$ is A^{-1}-close to $g_0(t)$ there. Hence on $\mathcal{C}_{s_0+\rho-A} \times [0, \sigma]$, $\bar{g}(\cdot)$ is A^{-1}-close to $g_0(\cdot)$, and in particular $R \leqslant 0$ there. We may then iterate this argument, applying Lemma 2.21 and the persistence theorem, $n = [T/\sigma]$ times and get that $\bar{g}(\cdot)$ is A^{-1}-close to $g_0(\cdot)$ on $\mathcal{C}_{s_0+n(\rho-A)} \times [0, T]$.

This finishes the proof of Theorem 2.19. □

3 Thick-thin decomposition theorem

Let (X, g) be a Riemannian 3-manifold and ε be a positive number. The *ε-thin part* of (X, g) is the subset $X^-(\varepsilon)$ of points $x \in X$ for which there exists $\rho \in (0, 1]$ such that on the ball $B(x, \rho)$ all sectional curvatures are at least $-\rho^{-2}$ and the volume of this ball is less than $\varepsilon\rho^3$. Its complement is called the *ε-thick part* of (X, g) and denoted by $X^+(\varepsilon)$. The aim of the section is to gather curvature and convergence estimates on the ε-thick part of $(M, t^{-1})g(t)$ as $t \to \infty$, when M is nonspherical, irreducible, and $g(\cdot)$ is a Ricci flow with $r(\cdot), \delta(\cdot)$-bubbling for suitably chosen surgery parameters. We will assume also that M does not have a metric with $\mathrm{Rm} \geq 0$: Otherwise the manifold would be, using Cheeger-Gromoll theorem, diffeomorphic to $\mathbf{R}^3$, $S^1 \times \mathbf{R}^2$, $T^2 \times \mathbf{R}$ or $K^2 \tilde{\times} \mathbf{R}$ (cf Appendix B2 in [BBB+10]), hence Seifert fibred.

Recall that $r(\cdot)$ has been fixed in Definition 2.16. In [BBB+10] Definition 11.1.4 we define a positive nonincreasing function $\bar{\delta}(\cdot)$ such that any Ricci flow with $r(\cdot), \delta(\cdot)$-bubbling-off satisfies some technical theorems—Theorems 11.1.3 and 11.1.6, analoguous to [Per03, Propositions 6.3 and 6.8]—if $\delta \leq \bar{\delta}$ and the initial metric is normalised.

Both Theorems 11.1.3 and 11.1.6 remain true for a Ricci flow with $r(\cdot), \delta(\cdot)$-bubbling-off on a noncompact non spherical irreducible manifold, with the weaker assumption that the metric has normalised curvature at time 0, i.e. $\operatorname{tr}\mathrm{Rm}^2 \leqslant 1$ for the initial metric, instead of being normalised in the sense of Definition 2.14. In particular it applies to metrics which are cusp-like at infinity. Indeed, the proofs of theorems 11.1.3 and 11.1.6 do not use the assumption on the volume of unit balls for the initial metric; it uses only the assumption on the curvature, mainly through the estimates (5)-(6). It uses neither the compactness of the manifold, the finiteness of the volume nor the particular manifold. We recall that the bulk of Theorem 11.1.3 is to obtain κ-non collapsing property, canonical neighbourhoods and curvature controls relatively to a distant ball satisfying a lower volume bound assumption. The parameters then depend of the distance to the ball and of its volume, not on time or initial data. These estimates are then used to control the thick part

(Theorem 11.1.6).

We assume that M does not have a metric with $\mathrm{Rm} \geq 0$. Given a Ricci flow with bubbling-off on M, we define

$$\rho(x,t) = \max\{\rho > 0 \ : \ \mathrm{Rm} \geqslant -\rho^{-2} \ \text{ on } B(x,t,\rho) \ \}$$

and $\rho_{\sqrt{t}} = \min\{\rho(x,t), \sqrt{t}\}$. For later purposes we state results similar to [Bam11a, propositions 4.1, 4.2], whose proofs follow from the analogues of [BBB$^+$10, 11.1.6 and 11.2.9]:

Proposition 3.1. *For every $w > 0$ there exists $\bar{\rho}(w) \in (0,1]$, $T = \bar{T}(w)$, $\bar{K} = \bar{K}(w) < \infty$ such that for any Ricci flow with $r(\cdot), \delta(\cdot)$-bubbling-off on M with $\delta(\cdot) \leq \bar{\delta}(\cdot)$ and normalised curvature at time 0, the following holds. For all $x \in M$ and $t \geqslant T$, if $\mathrm{vol}B(x,t,\rho_{\sqrt{t}}(x,t)) \geqslant w(\rho_{\sqrt{t}}(x,t))^3$ then $\rho_{\sqrt{t}}(x,t) \geqslant \bar{\rho}\sqrt{t}$ and $|\,\mathrm{Rm}\,| \leqslant \bar{K}t^{-1}$ on $B(x,t,\bar{\rho}\sqrt{t})$.*

We denote by $\tilde{M}$ the universal cover of M and $\tilde{g}(t)$ the lifted evolving metric, which is a Ricci flow with $r(\cdot), \delta(\cdot)$-bubbling-off if $g(t)$ is. If $x \in M$, we denote by $\tilde{x} \in \tilde{M}$ a lift of x and by $\tilde{B}(\tilde{x},t,r)$ the r-ball in $(\tilde{M}, \tilde{g}(t))$ centered at $\tilde{x}$. We have :

Proposition 3.2. *Under the same assumptions as in Proposition 3.1, if $\mathrm{vol}\tilde{B}(\tilde{x},t,\rho_{\sqrt{t}}(x,t)) \geq w(\rho_{\sqrt{t}}(x,t))^3$ then $\rho_{\sqrt{t}}(x,t) \geqslant \bar{\rho}\sqrt{t}$ and $|\,\mathrm{Rm}\,| \leqslant \bar{K}t^{-1}$ on $B(x,t,\bar{\rho}\sqrt{t})$.*

An evolving metric $\{g(t)\}_{t\in I}$ on M is said to have *finite volume* if $g(t)$ has finite volume for every $t \in I$. We denote this volume by $V(t)$. We shall need the following theorem, which says that as t becomes large, a Ricci flow with $r(\cdot), \delta(\cdot)$-bubbling-off and finite volume has a thick-thin decomposition as t goes to $+\infty$ (cf. [BBB$^+$10, Theorem 11.1.7].)

Theorem 3.3. *For every nonspherical, irreducible 3-manifold M and every complete, bounded curvature, finite volume Ricci flow with $(r(\cdot), \delta(\cdot))$-bubbling-off on M such that $\delta(\cdot) \leq \bar{\delta}(\cdot)$ and curvature is normalised at time 0, we have:*

i. There exists $C > 0$ such that $V(t) \leq Ct^{3/2}$.

ii. Let $w > 0$, $x_n \in M$ and $t_n \to +\infty$. If x_n is in the w-thick part of $(M, t_n^{-1}g(t_n))$ for every n, then the sequence of pointed manifolds $(M, t_n^{-1}g(t_n), x_n)$ subconverges smoothly to a complete finite volume pointed hyperbolic 3-manifold of sectional curvature $-1/4$.

iii. *For all $w > 0$, there exist $\bar{r} > 0$, $K(w)$ and $T(w) < \infty$ such that for all $x \in M$ and $t \geqslant T$, for all $r \in (0, \bar{r}]$, if $B(x,r) \subset (M, t^{-1}g(t))$ has $\mathrm{Rm} \geqslant -r^{-2}$ and $\mathrm{vol} B(x,r) \geqslant wr^3$ then $|\mathrm{Rm}| \leqslant Kr^{-2}$, $|\nabla \mathrm{Rm}| \leqslant Kr^{-3}$ and $|\nabla^2 \mathrm{Rm}| \leqslant Kr^{-4}$ on $B(x,r)$.*

Proof. Identical to Section 11.2 in [BBB$^+$10], using technical theorems 11.1.3 and 11.1.6. The assumption on the volume is used to prove that limits of rescaled parabolic neighbourhoods are hyperbolic (cf Proposition 11.2.3). □

Remark 3.4. In (iii) the conclusion holds on the full ball $B(x,r)$, see [BBB$^+$10, Remark 11.2.12].

Remark 3.5. The hypothesis that M is irreducible and nonspherical is not essential here, but since the definitions of canonical neighbourhoods we have given are adequate only in this case, it makes sense to keep those assumptions throughout.

For later purposes, namely to prove that cuspidal tori in the appearing hyperbolic pieces are incompressible in M, we need the following improvement of Theorem 3.3, which gives convergence of flows rather than metrics. With the notations of Theorem 3.3, we define $g_n := t_n^{-1}g(t_n)$ and $g_n(t) := t_n^{-1}g(tt_n)$, the latter being a Ricci flow with bubling-off such that $g_n(1) = g_n$. If g_{hyp} denotes the hyperbolic metric of sectional curvature $-1/4$, the Ricci flow $g_{\mathrm{hyp}}(t)$ satisfying $g_{hyp}(1) = g_{\mathrm{hyp}}$ is simply $g_{\mathrm{hyp}}(t) = tg_{\mathrm{hyp}}$. Consider $w > 0$, $t_n \to \infty$ and x_n in the w-thick part of (M, g_n). By Theorem 3.3 there exists a (sub)-sequence of (M, g_n, x_n) converging smoothly to $(H, g_{\mathrm{hyp}}, x_\infty)$. By relabeling, we can assume that the sequence converges. Then we have:

Theorem 3.6. *The sequence $(M \times [1,2], g_n(t), (x_n, 1))$ converges smoothly to $(H \times [1,2], g_{\mathrm{hyp}}(t), (x_\infty, 1))$.*

Proof. We need to show that, for all $A > 0$, for all n large enough, the rescaled parabolic ball $B(\bar{x}_n, 1, A) \times [1,2]$ is A^{-1}-close to $B(x_\infty, 1, A) \times [1,2]$. In what follows we but a bar on x_n to indicate that the ball is w.r.t $g_n(t)$.

We use the persistence theorem [BBB$^+$10, theorem 8.1.3], the hyperbolic limit $(H \times [1,2], g_{\mathrm{hyp}}(t), (x_n, 1))$ being the model $\mathcal{M}_0$ in the sense of [BBB$^+$10] p94. Fix $A > 0$ and let $\rho := \rho(\mathcal{M}_0, A, 1)$ be the parameter from the persistence theorem. By definition of $(H, g_{\mathrm{hyp}}, x_\infty)$, note that $(B(\bar{x}_n, 1, \rho), g_n)$ is ρ^{-1}-close to $(B(x_\infty, 1, \rho), g_{\mathrm{hyp}})$ for all large n, satisfying assumption (ii) of [BBB$^+$10, theorem 8.1.3]. To verify the other assumptions, we consider for each n, $\Delta_n \in [1,2]$ maximal such that

(i) $B(x_n, t_n, \rho\sqrt{t_n}) \times [t_n, \Delta_n t_n]$ is unscathed,

(ii) $|2t\,\mathrm{Ric}(x,t)+g(x,t)|_{g(t)} \leqslant 10^{-6}$ there.

The case $\Delta_n = 1$, where t_n is a singular time and a surgery affects the ball just at that time, is not *a priori* excluded.

Note that (ii) implies $|\,\mathrm{Rm}_{g_n}| \leqslant 1$ on the neighbourhood: one has $\mathrm{Ric}_{g(t)} \approx -\frac{1}{2t}g(t)$, or $\mathrm{Ric}_{g(tt_n)} \approx -\frac{1}{2tt_n}g(tt_n)$ for $t \in [1,\Delta_n]$, and then $\mathrm{Ric}_{g_n(t)} = \mathrm{Ric}_{t_n^{-1}g(tt_n)} \approx -\frac{1}{2tt_n}g(tt_n) = -\frac{1}{2t}g_n(t)$. Thus the sectional curvatures of $g_n(t)$ remain close to the interval $[-\frac{1}{4},-\frac{1}{8}]$.

The assumptions of [BBB$^+$10, theorem 8.1.3] being satisfied on $B(\bar{x}_n,1,\rho)\times[1,\Delta_n]$, the conclusion holds on $B(\bar{x}_n,1,A)\times[1,\Delta_n]$, that is $(B(\bar{x}_n,1,A)\times[1,\Delta_n],g_n(t))$ is A^{-1}-close to $(B(x_\infty,1,A)\times[1,\Delta_n],g_{\mathrm{hyp}}(t))$. We claim that $\Delta_n = 2$ for all n large enough.

Claim 2. For all n large enough, $\Delta_n = 2$.

Proof of claim. Let us put $t'_n := \Delta_n t_n$.

We prove first that there are at most finitely many integers n such that t'_n is a singular time where $B(x_n,t_n,\rho\sqrt{t_n})$ is scathed, that is $g_+(x,t'_n) \neq g(x,t'_n)$ for some $x \in B(x_n,t_n,\rho\sqrt{t_n})$.

Let us first describe the idea of the proof. Assume that t'_n is such a singular time. By definition of our (r,δ)-surgery, there is a 3-ball $B \ni x$, on which $g_+(t'_n) < g(t'_n)$, and whose boundary is the middle sphere of a strong δ-neck in $(M,g(t'_n))$, with scalar curvature $\approx h(t'_n)^{-2} >> 0$. By assumption (ii) above, $R<0$ at time t'_n on $B(x_n,t_n,\rho\sqrt{t_n})$, hence the middle sphere of the δ-neck cannot intersect $B(x_n,t_n,\rho\sqrt{t_n})$. It follows that $B(x_n,t_n,\rho\sqrt{t_n}) \subset B$, which is an almost standard cap for $g_+(t'_n)$. On the other hand, before the metric surgery the conclusion of the persistence theorem implies that $(B(x_n,t_n,A\sqrt{t_n}),g(t'_n))$ is almost homothetic to a (large) piece of the hyperbolic manifold H. Hence the surgery shrinks a piece of almost hyperbolic metric to a small standard cap, decreasing volume by a definite amount. As moreover $t^{-1}g(t)$ is volume decreasing along time, one can show that volume goes to zero if there are infinitely many such singular times, contradicting the assumption that H exists. We now go into details.

Let $\mu_0 > 0$ be the infimum of volumes of hyperbolic 3-manifolds (here with sectional curvature equal to $-1/4$). Fix $K \subset H$ a compact core with volume $> \frac{\mu_0}{2}$. For any $t \geqslant 1$ we then have $\mathrm{vol}_{g_{\mathrm{hyp}}(t)}(K) = t^{3/2}\mathrm{vol}_{g_{hyp}}(K) \geqslant t^{3/2}\frac{\mu_0}{2}$. We can assume A large enough such that $K \subset B(x_\infty,1,A)$. Then by closeness at time Δ_n:

$$\mathrm{vol}_{g_n(\Delta_n)}(B(x_n,1,A)) \geqslant \frac{1}{2}\mathrm{vol}_{g_{\mathrm{hyp}}(\Delta_n)}(B(x_\infty,1,A)) > \Delta_n^{3/2}\frac{\mu_0}{4}.$$

If B denotes the 3-ball containing $B(x_n, t_n, \rho\sqrt{t_n})$, then we also have $\mathrm{vol}_{g_n(\Delta_n)}(B) > \Delta_n^{3/2}\frac{\mu_0}{4}$. On the other hand, $\mathrm{vol}_{g_+(t'_n)}(B)$ is comparable to $h^3(t'_n)$, where $h(t'_n)$ is the cutoff parameter. Let us set $\bar{g}(t) := t^{-1}g(t)$. Computing the volumes with the metric $\bar{g}(t'_n) = {t'_n}^{-1}g(t'_n) = {t'_n}^{-1}t_n g_n(\Delta_n)$ gives :

$$\begin{aligned}\mathrm{vol}_{\bar{g}_+(t'_n)}(B) - \mathrm{vol}_{\bar{g}(t'_n)}(B) &= (t'_n)^{-3/2}\mathrm{vol}_{g_+(t'_n)}(B) - (t'_n)^{-3/2}t_n^{3/2}\mathrm{vol}_{g_n(\Delta_n)}(B) \\ &\leqslant (t'_n)^{-3/2}c.h^3(t'_n) - \frac{\mu_0}{4} < -\frac{\mu_0}{5}.\end{aligned}$$

Since $g_+(t) \leqslant g(t)$ on the whole manifold, we have

$$\mathrm{vol}_{\bar{g}_+(t'_n)}(M) - \mathrm{vol}_{\bar{g}(t'_n)}(M) < -\frac{\mu_0}{5}.$$

Now the proof of [BBB+10, Proposition 11.2.1] shows that $(t+\frac{1}{4})^{-1}g(t)$ is volume non-increasing along the Ricci flow. Since $g_+ \leqslant g$ at singular times, this monotonicity holds for a Ricci flow with bubbling-off. One easily deduces by comparing the $(t+1/4)^{-1}$ and the t^{-1} scaling that $\bar{g}(t)$ is volume decreasing, and more precisely that for all $t' > t$:

$$\mathrm{vol}_{\bar{g}(t')}(M) \leqslant \left(\frac{t'+1/4}{t'}\frac{t}{t+1/4}\right)^{3/2}\mathrm{vol}_{\bar{g}(t)}(M) < \mathrm{vol}_{\bar{g}(t)}(M).$$

It particular, the sequence $\mathrm{vol}_{g_n}(M) = \mathrm{vol}_{\bar{g}(t_n)}(M)$ is decreasing. Moreover, if $[t_n, t_m]$ contains a singular time t'_n as above, then

$$\begin{aligned}\mathrm{vol}_{g_m}(M) = \mathrm{vol}_{\bar{g}(t_m)}(M) &\leqslant \mathrm{vol}_{\bar{g}_+(t'_n)}(M) \\ &< \mathrm{vol}_{\bar{g}(t'_n)}(M) - \frac{\mu_0}{5} \\ &\leqslant \mathrm{vol}_{\bar{g}(t_n)}(M) - \frac{\mu_0}{5} \\ &= \mathrm{vol}_{g_n}(M) - \frac{\mu_0}{5}.\end{aligned}$$

On the other hand, $\mathrm{vol}_{g_n}(M) \geqslant \mathrm{vol}_{g_{\mathrm{hyp}}}(H) \geqslant \mu_0$. Thus there are at most finitely many such singular times. We conclude that $B(x_n, t_n, \rho\sqrt{t_n})$ is unscathed at time t'_n for all n large enough.

From now we suppose n large enough such that this property holds. Recall that singular times form a discrete subset of $\mathbf{R}$, hence there are some small $\sigma_n > 0$ such that $B(x_n, t_n, \rho\sqrt{t_n})$ is unscathed on $[t_n, t'_n + \sigma_n]$. By maximality of Δ_n, when $\Delta_n < 2$ we must have $|2t\,\mathrm{Ric}(x,t) + g(x,t)|_{g(t)} = 10^{-6}$ at time t'_n for some $x \in \overline{B(x_n, t_n, \rho\sqrt{t_n})}$. Otherwise by continuity we choose find σ_n small enough such that (ii) holds on $[t_n, t'_n + \sigma_n] \subset [t_n, 2t_n]$, contradicting

the maximality of Δ_n.

We now show that for all large n, $|2t\operatorname{Ric}(x,t)+g(x,t)|_{g(t)} < 10^{-6}$ at time t'_n on $\overline{B(x_n,t_n,\rho\sqrt{t_n})}$, which will imply that $\Delta_n = 2$ by the discussion above. The estimate will follow from [BBB+10, Proposition 11.2.7]. Let $r > 0$ be a small radius ($r < 1/10 < A$) and $w' > 0$ be such that $\mathrm{vol}_{g_{\mathrm{hyp}}}(B(x_\infty,1,r) \geqslant w'r^3$. As $g_{\mathrm{hyp}}(t) = tg_{\mathrm{hyp}}$, we have $B(x_\infty,t,r\sqrt{t}) = B(x_\infty,1,r)$, and $\mathrm{vol}_{g_{\mathrm{hyp}}(t)}(B(x_\infty,t,r\sqrt{t})) \geqslant t^{3/2}w'r^3$. By closeness at time Δ_n on $B(\bar{x}_n,1,A) \supset B(\bar{x}_n,1,r)$ between $g_n(\Delta_n)$ and $g_{\mathrm{hyp}}(\Delta_n)$, we deduce

$$\mathrm{vol}_{g(t'_n)}(B(x_n,t'_n,r\sqrt{t'_n})) \geqslant \frac{1}{2}(t_n)^{3/2}(\Delta_n)^{3/2}w'r^3 = \frac{1}{2}w'(r\sqrt{t'_n})^3,$$

verifying assumption (i) of [BBB+10, Proposition 11.2.7]. As $B(x_n,t'_n,r\sqrt{t'_n}) \subset B(x_n,t_n,A\sqrt{t_n})$, we also have $\operatorname{Ric} \approx -\frac{1}{2t'_n}g(t'_n)$ there, by property (ii) above. Hence $\operatorname{Rm} \geqslant -(r\sqrt{t'_n})^{-2}$ and assumption (ii) of [BBB+10, Proposition 11.2.7] is also satisfied. Now we choose $A' > 0$ large enough such that $A'r > 10\rho$. The proposition applied with parameters $w'/2, r, \xi = 10^{-6}$ and A' gives T such that for t'_n larger than T, $g(t'_n)$ is ξ-almost hyperbolic on $B(x_n,t'_n,A'r\sqrt{t'_n})$.

There remains to show that

$$B(x_n,t'_n,A'r\sqrt{t'_n}) \supset \overline{B(x_n,t_n,\rho\sqrt{t_n})}.$$

On $B(x_n,t_n,\rho\sqrt{t_n}) \times [t_n,t'_n]$, we have $\operatorname{Ric} \approx -\frac{1}{2t}g \geqslant -\frac{1}{t}g$. Then $g' = -2\operatorname{Ric}_g \leqslant \frac{2}{t}g(t)$, and integration gives $g(t'_n) \leqslant (\frac{t'_n}{t_n})^2 g(t_n) \leqslant 4g(t_n)$ on the ball $B(x_n,t_n,\rho\sqrt{t_n})$. Thus, if $d_{t_n}(x_n,x) \leqslant \rho\sqrt{t_n}$, then $d_{t'_n}(x_n,x) \leqslant 2\rho\sqrt{t_n} < A'r\sqrt{t_n}$. This proves the required inclusion and we conclude that the inequality in property (ii) is strict on the closure of the ball. Together with the first part of the proof and the maximality of Δ_n this implies that $\Delta_n = 2$ for n large enough, proving the claim. ☐

As already noted, we then have, by the persistence theorem, that $B(x_n,1,A)\times[1,2]$, with the rescaled flow $g_n(t)$, is A^{-1}-close to $B(x_\infty,1,A)\times[1,2]$ for all n large enough. This concludes the proof of Theorem 3.6. ☐

From theorem 3.6 one easily obtain:

Corollary 3.7. *Given $w > 0$ there exist a number $T > 0$ and a nonincreasing function $\beta : [T,+\infty) \to (0,+\infty)$ tending to 0 at $+\infty$ such that if (x,t) is in the w-thick part of $(M,t^{-1}g(t))$ with $t \geq T$, then there exists a pointed hyperbolic manifold $(H,g_{\mathrm{hyp}},*)$ such that:*

(i) $P(x,t,\beta(t)^{-1}\sqrt{t},t)$ *is* $\beta(t)$*-homothetic to* $P(*,1,\beta(t)^{-1},1) \subset H \times [1,2]$*, endowed with* $g_{\text{hyp}}(s) = s g_{\text{hyp}}(1)$*),*

(ii) For all $y \in B(x,t,\beta(t)^{-1}\sqrt{t})$ *and* $s \in [t,2t]$*,*

$$|\bar{g}(y,s) - \bar{g}(y,t)| < \beta,$$

where the norm is in the $C^{\beta^{-1}}$*-topology for the metric* $\bar{g}(t) = t^{-1}g(t)$*.*

Remark 3.8. Point (ii) of the corollary will be useful to extend on parabolic neighbourhoods an approximation given on balls.

4 Incompressibility of the boundary tori

We prove that under the hypotheses of the previous section the tori that separate the thick part from the thin part are incompressible.

More precisely, we consider a nonspherical, irreducible, complete, bounded curvature, finite volume Ricci flow with $(r(\cdot),\delta(\cdot))$-bubbling-off $g(\cdot)$ on M such that $\delta(\cdot) \leq \bar{\delta}(\cdot)$, *and whose universal cover has bounded geometry.* We call *hyperbolic limit* a pointed 'hyperbolic' manifold of finite volume and sectional curvature $= -1/4$ that appears as a pointed limit of $(M, t_n^{-1}g(t_n), x_n)$ for some sequence $t_n \to \infty$. **In this section, we assume the existence of at least one hyperbolic limit** $(H, g_{\text{hyp}}, *)$**, which is supposed not closed.**

Given a hyperbolic limit H, we call *compact core of* H, a compact submanifold $\bar{H} \subset H$ whose complement consists of finitely many product neighbourhoods of the cusps. Then for large n, we have an approximating embedding $f_n : \bar{H} \to M$ which is almost isometric with respect to the metrics g_{hyp} and $t_n^{-1}g(t_n)$. The goal of this section is to prove the following result:

Proposition 4.1. *If* n *is large enough, then for each component* T *of* $\partial\bar{H}$*, the image* $f_n(T)$ *is incompressible in* M*.*

We argue following Hamilton's nonsingular paper [Ham99]. A key tool is the stability of the hyperbolic limit H: it is a limit along the flow, not just along a sequence of times. We give a statement after Kleiner-Lott (cf. [KL08, Proposition 90.1].)

Proposition 4.2 (Stability of thick part)**.** *There exist a number* $T_0 > 0$*, a nonincreasing function* $\alpha : [T_0, +\infty) \to (0, +\infty)$ *tending to* 0 *at* $+\infty$*, a finite*

collection $\{(H_1, *_1), \ldots, (H_k, *_k)\}$ *of hyperbolic limits and a smooth family of smooth maps*

$$f(t) : B_t = \bigcup_{i=1}^{k} B(*_i, \alpha(t)^{-1}) \to M$$

defined for $t \in [T_0, +\infty)$*, such that*

(i) *The* $C^{\alpha(t)^{-1}}$*-norm of* $t^{-1}f(t)^*g(t) - g_{\text{hyp}}$ *is less than* $\alpha(t)$*;*

(ii) *For every* $t_0 \geq T_0$ *and every* $x_0 \in B_{t_0}$*, the time-derivative at* t_0 *of the function* $t \mapsto f(t)(x_0)$ *is less than* $\alpha(t_0)t_0^{-1/2}$*.*

(iii) $f(t)$ *parametrizes more and more of the thick part: the* $\alpha(t)$*-thick part of* $(M, t^{-1}g(t))$ *is contained in* $\mathrm{im}(f(t))$*.*

Remark 4.3. Any hyperbolic limit H is isometric to one of the H_i. Indeed, let $* \in H$ and $w > 0$ such that $* \in H^+(w)$. Then x_n is in the $w/2$-thick part of $(M, t_n^{-1}g(t_n))$ for n large enough. Assume that $f(t_n)^{-1}(x_n) \in B(*_i, \alpha(t_n)^{-1})$ for a subsequence. Then $f(t_n)^{-1}(x_n)$ remains at bounded distance of $*_i$, otherwise it would go into a cusp contradicting the $w/2$-thickness of x_n. It follows that (M, x_n) and $(M, f(t_n)(*_i))$ will have the same limit, up to an isometry.

4.1 Proof of Proposition 4.2

We give a proof after Kleiner-Lott [KL08].

Definition 4.4. Given pointed Riemannian manifolds (X, x) and (Y, y) and $\varepsilon > 0$, we say that (X, x) is *ε-close* to (Y, y) is there is a pointed map $f : (X, x) \to (Y, y)$ such that

$$f_{|\overline{B(x,\varepsilon^{-1})}} : \overline{B(x, \varepsilon^{-1})} \to Y$$

is a diffeomorphism onto its image and such that

$$|f^*g_Y - g_X| < \varepsilon$$

in the $C^{\varepsilon^{-1}}$ topology on $\overline{B(x, \varepsilon^{-1})}$. We call such a map f an *ε-approximation.*

Note that nothing is required on the complement of $\overline{B(x, \varepsilon^{-1})}$. We will make the abuse of langage of calling a partially defined map from $W \subset X$ to Y an ε-approximation if it is defined on $\overline{B(x, \varepsilon^{-1})} \subset W$ and satisfies the properties above.

We recall a fact from hyperbolic geometry:

Lemma 4.5. *Let (X, x) be a pointed hyperbolic 3-manifold of finite volume. Then for each $\zeta > 0$ there exists $\xi > 0$ such that if X' is a hyperbolic 3-manifold of finite volume with at least as many cusps as X, and $f : (X, x) \to X'$ is a ξ-approximation, then there is an isometry from X to X' which is ζ-close to f.*

Proof. [KL08, Lemma 90.11] $\square$

To prove Proposition 4.2, we construct the familly $\{(H_1, *_1), \dots, (H_k, *_k))$ inductively. Recall that there exists $w_0 > 0$ such that any complete finite volume hyperbolic 3-manifold has a w_0-thick point. Let H_1 be a hyperbolic limit with the fewest number of cusps, and fix $*_1 \in H_1^+(w_0)$ a w_0-thick point. We set $w_1 := w_0/2$ and $\bar{M}_t := (M, t^{-1}g(t))$.

The first step is to construct a familly $\{f_0(t) | t \geqslant T_0\}$ of δ-approximations from $(H_1, *_1)$ to $(\bar{M}_t, f_0(t)(*_1))$. This is done by induction: one defines for each $j \geqslant 0$, $f_0(t) = f_0(2^j T_0)$ on $[2^j T_0, 2^{j+1} T_0[$ such that

- $f_0(2^j T_0)(*_1) =: \bar{x}_j \in \bar{M}_{2^j T_0}(w_1)$,
- $f_0(2^j T_0)$ is a $2\beta(2^j T_0)$-approximation between $(H_1, *_1)$ and $(\bar{M}_{2^j T_0}, f_0(2^j T_0)(*_1))$.

Here β is the parameter from Corollary 3.7 and depends on w_1. Moreover, the maps $f_0(t)$ and $f_0(2^{j+1}T_0)$ will be be uniformly close when $t < 2^{j+1}T_0$ is close to $2^{j+1}T_0$. Then using part (ii) of the corollary and a standard smoothing argument, one obtain a smooth familly $\{f_1(t), | t \geqslant T_0\}$ satisfying (i)-(ii) of Proposition 4.2.

We begin the construction of $\{f_0(t)\}$. Note that for δ sufficiently small, if $f_0(t)$ is a δ-approximation as above then $f_0(t)(*_1) \in \bar{M}_t^+(w_1)$. Let $\xi_1, \xi_2, \xi_3, \xi_4$ be positive parameters to be specified later. Assume T_0 is large enough so that $2\beta(T_0) < \xi_1$. By definition of $(H_1, *_1)$, for $T_0 = t_n$ large enough depending on ξ_1, there is a $2\xi_1$-approximation $\Phi_0 : (H_1, *_1) \to (\bar{M}_{t_n}, x_n)$. We define the $2\beta(T_0)$-approximation $f_0(T_0)$ as follows. Note that $x_n \in \bar{M}_{T_0}^+(w_1)$. By part (i) of Corollary 3.7 there exists a hyperbolic manifold $(H', *')$ and a $\beta(T_0)$-approximation $\psi : (H', *') \to (\bar{M}_{t_n}, x_n)$. Then $\psi^{-1} \circ \Phi_0 : (H_1, *_1) \to (H', *')$ is a ξ_2-approximation, if ξ_1 is small enough depending on ξ_2. By Lemma 4.5, there exists an isometry $I : H_1 \to H'$, ξ_3-close to $\psi^{-1} \circ \Phi_0$, if ξ_2 is sufficiently small depending on ξ_3 and H_1. Here we use the fact that H_1 has the least possible number of cusps. Then we set $f_0(T_0) := \psi \circ I$ and $\bar{x}_0 = f_0(T_0)(*_1)$. As ψ is a $\beta(T_0)$-approximation and I is an isometry which moves $*_1$ at distance at most ξ_3 from $*'$, by reducing the radius of the ball we obtain that $f_0(T_0)$ is a $2\beta(T_0)$-approximation from $(H_1, *_1)$ to $(\bar{M}_{T_0}, \bar{x}_0)$, if $\beta(T_0)$ is small enough compared to ξ_3. Note that $f(T_0)$ is ξ_4-close to the

initial approximation Φ_0, is ξ_3 is small enough depending on ξ_4. One then defines, for $t \in [T_0, 2T_0[$, $f_0(t) := f_0(T_0) : B(*, (2\beta(T_0))^{-1}) \to \bar{M}_t$ and we set $\Phi_1 := f_0(T_0) : B(*_1, (2\beta(T_0))^{-1}) \to \bar{M}_{2T_0}$. By part (ii) of Corollary 3.7, both $f_0(t)$ and Φ_1 are $2\xi_1$-approximations, if $\beta(T_0)$ is small enough compared to ξ_1, that is if T_0 is sufficiently large. As Φ_1 is a $2\xi_1$-approximation, $\bar{x}_0 \in \bar{M}^+_{2T_0}(w_1)$. One can then iterate the procedure above on intervals $[2^jT_0, 2^{j+1}T_0[$, for $j \geqslant 1$, starting from $(\Phi_j, \bar{x}_{j-1})$ at time 2^jT_0.

Having constructed $2\xi_1$-approximations $\{f_0(t)|t \geqslant T_0\}$, which are $2\beta(2^jT_0)$-approximations at time 2^jT_0, one can use again part (ii) of Corollary 3.7 to find some $\alpha(t) : [T_0, +\infty) \to (0, +\infty)$, decreasing to zero at infinity, such that $f(t)$ are $\alpha(t)$-approximations. By a standard smoothing argument, noticing that jumps from $\{f_0(t)|2^jT_0 \leqslant t < 2^{j+1}T_0\}$ to $f_0(2^{j+1}T_0)$ are controlled by ξ_4, one can convert $\{f_0(t)\}$ into a smooth familly $\{f_1(t)\}$ satisfying properties (i)-(ii) of Proposition 4.2.

If for all $w > 0$, $\bar{M}^+_t(w)$ is contained in $\mathrm{im}(f_1(t))$ for all t large enough, one can ajust the function $\alpha(t)$ so that Assertion (iii) of the proposition is also satisfied. If not, there is $w > 0$ and a sequence (x_n, t_n) with $t_n \to \infty$ such that $x_n \in \bar{M}^+_{t_n}(w)$ but $x_n \notin \mathrm{im}(f_1(t_n))$. Fix such $w > 0$ and let H_2 be a hyperbolic limit with the fewest number of cusps among all hyperbolic limits obtained from such sequences (x_n, t_n), and pick $*_2 \in H_2^+(w_0)$. We can repeat the construction above to obtain a smooth familly $\{f_2(t)\}$ approximating $(H_2, *_2)$, satisfying assertions (i)-(ii) of the proposition, and such that $\mathrm{im}(f_2(t))$ is disjoint from $\mathrm{im}(f_1(t))$. As long as we cannot obtain (iii) we iterate this procedure. Now recall that $t \to \mathrm{vol}(\bar{M}_t)$ is bounded and that any hyperbolic 3-manifold has volume at least $\mu_0 > 0$. For t large enough, each $f(t)(B(*_i, \alpha(t)^{-1}))$ has volume at least $\mu_0/2$, hence the procedure above must stop after k steps. The desired function $\{f(t)\}$ is the union of $f_1(t), \ldots, f_k(t)$.

This concludes the proof of proposition 4.2.

4.2 Proof of Proposition 4.1

The proof of Hamilton [Ham99] is by contradiction. Assuming that some torus is compressible, one finds an embedded compressing disk for each time further. Using Meeks and Yau [MY80], the compressing disks can be chosen of least area. By controlling the rate of change of area of these disks, Hamilton shows that the area must go to zero in finite time—a contradiction.

Due to the possible noncompactness of our manifold, the existence of the least area compressing disks is not ensured: an area minimising sequence of disks can go deeper and deeper in an almost hyperbolic cusp. We will

tackle this difficulty by considering the universal cover, which has bounded geometry, when necessary.

Let us fix some notation. For all small $a > 0$ we denote by $\bar{H}_a$ the compact core in H whose boundary consists of horospherical tori of diameter a. By Proposition 4.2 and Remark 4.3, we can assume that the map $f(t)$ is defined on $B(*, \alpha(t)^{-1}) \supset \bar{H}_a$ for t larger than some $T_a > 0$. For all $t \geqslant T_a$ the image $f(t)(\bar{H}_a)$ is well defined and the compressibility in M of a given boundary torus $f(t)(\partial \bar{H}_a)$ does not depend on t or a. We assume that some torus T of $\partial \bar{H}_a$ has compressible image in M. Below we refine the choice of the torus T.

We define, for some fixed $a > 0$,

$$Y_t = f(t)(\bar{H}_a), \quad T_t = f(t)(T) \quad \text{and} \quad W_t = M - \operatorname{int}(Y_t)$$

Our first task is to find a torus in ∂Y_t which is compressible in W_t. Note that T_t is compressible in M, incompressible in Y_t which is the core of a hyperbolic 3-manifold, but not necessarily compressible in W_t: Y_t could be contained in a solid torus and T_t compressible on this side. Consider the surface $\partial Y_t \subset M$ (not necessarily connected). As the induced map $\pi_1(\partial Y_t) \to \pi_1(M)$ is non injective by assumption, Corollary 3.3 of Hatcher [Hat05] tells that there is a compressing disk $D \subset M$, with $\partial D \subset \partial Y_t$ homotopically non trivial and $\operatorname{int}(D) \subset M - \partial Y_t$. As $\operatorname{int}(D)$ is not contained in Y_t, one has $\operatorname{int}(D) \subset W_t$. Relabel T_t the connected component of ∂Y_t which contains ∂D and $T \subset \partial \bar{H}_a$ its $f(t)$-preimage. Then T_t is compressible in W_t.

Let X_t be the connected component of W_t which contains D. By [BBB+10, Lemma A.3.1] there are two exclusive possibilities:

(1) X_t is a solid torus. In this case, Meeks-Yau [MY80] provide a least area compressing disk $D_t^2 \subset X_t$ where $\partial D^2 \subset T_t$ is in a given nontrivial free homotopy class.

(2) Y_t is contained in a 3-ball B. Then Y_t lifts isometrically to a 3-ball in the universal cover $(\tilde{M}, \tilde{g}(t))$. Let $\tilde{Y}_t$ be a copy of Y_t in $\tilde{M}$. By [Hat05] again, there is a torus $\tilde{T}_t \subset \partial \tilde{Y}_t$ compressible in $\tilde{M} - \partial \tilde{Y}_t$, hence in $\tilde{M} - \tilde{Y}_t$. We denote by $\tilde{X}_t$ the connected component of $\tilde{M} - \operatorname{int}(\tilde{Y}_t)$ in which $\tilde{T}_t$ is compressible. As $(\tilde{M}, \tilde{g}(t))$ has bounded geometry, there is a compressing disk $D_t^2 \subset \tilde{X}_t$ of least area with $\partial D_t^2 \subset \tilde{T}_t$ in a given nontrivial free homotopy class.

We define a function $A : [T_a, +\infty) \to (0, +\infty)$ by letting $A(t)$ be the infimum of the areas of such embedded disks. Similarly to [KL08, Lemma 91.12] we have

Lemma 4.6. *For every $D > 0$, there is a number $a_0 > 0$ with the following property. Given $a \in (0, a_0)$ there exists $T'_a > 0$ such that for all $t_0 \geqslant T'_a$ there is a piecewise smooth function $\bar{A}$ defined in a neighbourhood of t_0 such that $\bar{A}(t_0) = A(t_0)$, $\bar{A} \geqslant A$ everywhere, and*

$$\bar{A}'(t_0) < \frac{3}{4}\left(\frac{1}{t_0 + \frac{1}{4}}\right) A(t_0) - 2\pi + D$$

if $\bar{A}$ is smooth at t_0, and $\lim_{t \to t_0, t > t_0} \bar{A}(t) \leqslant \bar{A}(t_0)$ if not.

Proof. The proof is similar to the proof of [KL08, Lemma 91.12], and somewhat simpler as we don't have topological surgeries. Recall that our Ricci flow with bubbling-off $g(t)$ is non increasing at singular times, hence the unscathedness of least area compressing disks ([KL08, Lemma 91.10]) is not needed: we have $\lim_{t \to t_0, t > t_0} A(t) \leqslant A(t_0)$ if t_0 is singular. However, something must be said about [KL08, Lemma 91.11]. This lemma asserts that given $D > 0$, there is $a_0 > 0$ such that for $a \in (0, a_0)$ and $T \subset H$ a horospherical torus of diameter a, for all t large enough $\int_{\partial D_t^2} \kappa_{\partial D_t^2} ds \leqslant \frac{D}{2}$ and $\mathrm{length}(\partial D_t^2) \leqslant \frac{D}{2}\sqrt{t}$, where $\kappa_{\partial D_t^2}$ is the geodesic curvature of ∂D_t^2. Its proof relies on the fact that an arbitrarily large collar neighbourhood of T_t in W_t is close (for the rescaled metric $t^{-1}g(t)$) to a hyperbolic cusp if t is large enough. In case (1) above, this holds on $X_t \cap f(t)B(*, \alpha(t)^{-1}))$ by Proposition 4.2. In case (2) observe that $f(t)(B(*, \alpha(t)^{-1}))$ is homotopically equivalent to the compact core $\bar{H}_t$, hence lifts isometrically to $(\tilde{M}, \tilde{g}(t))$. It follows that $\tilde{X}_t$ also has an arbitrarily large collar neighbourhood of $\tilde{T}_t$ close to a hyperbolic cusp.

The rest of the proof is identical to the proof of [KL08, Lemma 91.12] and hence omitted. ☐

Fix $D < 2\pi$, $a \in (0, a_0)$ and T'_a as in Lemma 4.6. Since A is locally bounded for $t \geqslant T'_a$ by Lemma 4.6, A is left continuous and lower semicontinuous from the right. Indeed if $t_k \to t$, the minimising compressing disks $D_{t_k}^2 \subset X_{t_k}$ will have uniformly bounded area in a space-time neighbourhood of bounded geometry, hence compactness theorems apply. They give a limit disk D^2 whose area for the limit metric, $g(t)$ or $g_+(t)$ in case (1), $\tilde{g}(t)$ or $\tilde{g}_+(t)$ in case (2), is the limit $\lim_{k \to \infty} \mathrm{area}(N_{t_k})$.

Then we consider the solution $\hat{A} : [T'_a, +\infty) \to \mathbf{R}$ of the ODE

$$\hat{A}' = \frac{3}{4}\left(\frac{1}{t + \frac{1}{4}}\right) \hat{A} - 2\pi + D$$

with initial condition $\hat{A}(T'_a) = A(T'_a)$. By a continuity argument, $A(t) \leqslant \hat{A}(t)$ for all $t \geqslant T'_a$. However,

$$\hat{A}(t)\left(t+\frac{1}{4}\right)^{-3/4} = 4(-2\pi + D)\left(t+\frac{1}{4}\right)^{1/4} + \text{const},$$

which implies that $\hat{A}(t) < 0$ for large t, contradicting the fact that $A(t) > 0$.

This finishes the proof of Proposition 4.2.

5 Collapsing theory

In this section we gather results describing the thin part of $(M, t^{-1}g(t))$, anologous to those of [MT08, theorems 0.2, 1.1] and [Bam11a, theorem 6.1].

Let (M_n, g_n) be a sequence of Riemannian 3-manifolds.

Definition 5.1. We say that g_n has *locally controlled curvature in the sense of Perelman* if for all $\varepsilon > 0$ there exist $\bar{r}(\varepsilon) > 0$, $K(\varepsilon) > 0$ such that for n large enough , if $0 < r \leq \bar{r}(\varepsilon)$, if $x \in (M_n, g_n)$ satisfies $\mathrm{vol}B(x,r) \geq \varepsilon r^3$ and $\sec \geq -r^{-2}$ on $B(x,r)$ then $|\operatorname{Rm}(x)| \leqslant Kr^{-2}$, $|\nabla \operatorname{Rm}(x)| \leqslant Kr^{-3}$ and $|\nabla^2 \operatorname{Rm}(x)| \leqslant Kr^{-4}$ on $B(x,r)$.

Remark 5.2. Note that if $g_n = {t_n}^{-1} g(t_n)$, where $g(\cdot)$ is as in Theorem 3.3 and $t_n \to \infty$, g_n has locally controlled curvature in the sense of Perelman.

Definition 5.3. We say that (g_n) *collapses* if there exists a sequence $\varepsilon_n \to 0$ of positive numbers such that (M_n, g_n) is ε_n-thin for all n.

Theorem 5.4. *Assume that (M_n, g_n) is a sequence of noncompact Riemannian oriented 3-manifold, irreducible and atoroidal, complete or with convex boundary, and $w_n \to 0$ is a sequence of positive numbers such that*

1) g_n is cusp-like at infinity for each n,

2) each boundary component of ∂M_n is an incompressible torus of diameter at most w_n and with a topologically trivial collar neighbourhood containing all points within distance 1 of the boundary on which the sectional curvatures are between $-5/16$ and $-3/16$,

3) (g_n) is w_n-thin,

4) (g_n) has locally controlled curvature in the sense of Perelman,

then for all n large enough M_n is a graph manifold.

The proof of theorem 5.4 follows, as in [MT08, Section 1.2] from another topological result (compare with [MT08, Theorem 1.1]):

Theorem 5.5. *Under the assumptions of theorem 5.4, for all* n *large enough there are compact, codimension-0, smooth submanifolds* $V_{n,1} \subset M_n$ *and* $V_{n,2} \subset M_n$ *with* $\partial M_n \subset V_{n,1}$*, and a closed (as subset) submanifold* $V_{n,1,cusp} \subset M_n$ *with the following properties.*

1. *Each connected component of* $V_{n,1}$ *is diffeomorphic to one of the following:*

   (a) *a* T^2*-bundle over* S^1 *or a union of two twisted* I*-bundles over the Klein bottle along their common boundary;*

   (b) $T^2 \times I$ *or* $S^2 \times I$*, where* I *is a closed interval;*

   (c) *a compact* 3*-ball or the complement of an open* 3*-ball in* $\mathbf{R}P^3$*;*

   (d) *a twisted* I*-bundle over the Klein bottle or a solid torus.*

1'. *Each connected component of* $V_{n,1,cusp}$ *is diffeomorphic to* $T^2 \times [0, \infty)$ *and is a neighbourhood of a cuspidal end of* M_n*.*

2'. $V_{n,1} \cap V_{n,1,cusp} = \emptyset$*,* $V_{n,2} \cap V_{n,1} = \partial V_{n,2} \cap \partial V_{n,1}$ *and* $V_{n,2} \cap V_{n,1,cusp} = \partial V_{n,2} \cap \partial V_{n,1,cusp}$*.*

3. *If* X_0 *is a* 2*-torus component of* $\partial V_{n,1}$ *then* $X_0 \subset \partial V_{n,2}$ *if and only if* X_0 *is not a boundary component of* M_n*.*

4. *If* X_0 *is a* 2*-sphere component of* $\partial V_{n,1}$ *then* $X_0 \cap \partial V_{n,2}$ *is diffeomorphic to an annulus.*

5'. $V_{n,2}$ *is the total space of a locally trivial* S^1*-bundle and* $\partial V_{n,1} \cap \partial V_{n,2}$ *and* $\partial V_{n,1,cusp} \cap \partial V_{n,2}$ *are saturated under this fibration.*

6'. $M_n \setminus (V_{n,1} \cup V_{n,2} \cup V_{n,1,cusp})$ *is a disjoint union of solid tori and solid cylinders, i.e. copies of* $D^2 \times I$*. The boundary of each solid torus is a boundary component of* $V_{n,2}$ *and each solid cylinder* $D^2 \times I$ *meets* $V_{n,1}$ *exactly in* $D^2 \times \partial I$*.*

Proof of theorem 5.5. Roughly speaking, $V_{n,1}$ and $V_{n,1,cusp}$ are subsets of M_n Gromov-Hausdorff close to 1-dimensional spaces and $V_{n,2}$ is close to 2-dimensional Alexandrov spaces. Let us explain briefly where the proof of [MT08, Theorem 1.1] has to be modified to prove Theorem 5.5. It suffices to replace Proposition 5.2 of [MT08] by the following.

Proposition 5.6. *Consider the subset $X_{n,1} \subset M_n$ consisting of all points $x \in M_n$ for which $B_{g'_n(x)}(x, 1)$ is within $\hat{\varepsilon}$ of a standard 1-dimensional ball J and the distance from x to the endpoints (if any) of J is at least $1/50$. Then there are two disjoint subsets $U_{n,1}, U_{n,1,cusp} \subset M_n$ containing $X_{n,1}$ with the following properties:*

1. *Each set $U_{n,1}$ is relatively compact and each component of $U_{n,1}$ is either a 2-torus bundle over the circle, or diffeomorphic to a product of either S^2 or T^2 with an open interval.*

1'. *Each connected component of $U_{n,1,cusp}$ is diffeomorphic to $T^2 \times (0, +\infty)$ and is a neighbourhood of a cuspidal end of M_n (if any)*

2. *Let $\mathcal{E}$ be a non-compact end of $U_{n,1}$ or a non-cuspidal end of $U_{n,1,cusp}$, then there is a point $x_{\mathcal{E}} \in X_{n,1}$, and an interval product structure centered at $x_{\mathcal{E}}$ with ε'-control, $p_{x_{\mathcal{E}}} : U(x_{\mathcal{E}}) \to J(x_{\mathcal{E}})$, where $J(x_{\mathcal{E}})$ is an interval of length $\geq 1/100$, with the property that $U(x_{\mathcal{E}})$ is a neighbourhood of the end $\mathcal{E}$.*

3. *Let $\mathcal{E}$ (resp. $\mathcal{E}'$) be a non-compact end of $U_{n,1}$ or a non-cuspidal end of $U_{n,1,cusp}$, then the neighbourhoods $U(x_{\mathcal{E}})$ and $U(x_{\mathcal{E}'})$ are disjoint.*

4. *For each point $x \in X_{n,1}$, the ball $B_{g'_n(x)}(x, 1/400)$ is contained in $U_{n,1}$ or in $U_{n,1,cusp}$.*

The proof is the same as for Proposition 5.2 of [MT08], components $U_{n,1,cusp}$ coming from the assumption that g_n is cusp-like at infinity. Then the construction of the sets $V_{n,1}$, $V_{n,2}$ and $V_{n,1,cusp}$ follows section 5 of [MT08], the components of $V_{n,1,cusp}$ being obtained from $U_{n,1,cusp}$ in the same manner that one obtains $V_{n,1}$ from $U_{n,1}$. □

In the next section we consider a sequence $M_n = (M, g_n)$ of irreducible atoroidal 3-manifolds with cusp-like metrics. For all n large enough, M_n has a decomposition $M_n = H_n \cup G_n$, where G_n collapses as in Theorem 5.4 and H_n is, when non empty, diffeomorphic to a hyperbolic manifold of finite volume, with $\bar{H}_n \cap G_n$ being a finite union of incompressible tori of M. In this case the atoroidality assumption on M implies that each connected component of G_n is diffeomorphic to $T^2 \times [0, +\infty)$, hence theorem 5.4 is not needed to conclude that G_n is a graph manifold. However, to obtain the last conclusion of Theorem 1.2, that is that $t \mapsto g(t)$ is smooth for large t, it is necessary to have a precise description of the collapsing on $T^2 \times [0, +\infty)$. Below we state a proposition analogous to [Bam11a, Proposition 6.1], in the very special case

of the manifold $T^2 \times [0, +\infty)$. This result can obtained from Theorem 5.5 as in [Bam11a, section 6]. We recall that

$$\rho(x) = \max\{r > 0, \mathrm{Rm} \geqslant -r^{-2} \text{ on } B(x, r)\}$$

and we set $\rho_1 = \min(\rho, 1)$.

Proposition 5.7. *For every two continuous functions $\bar{r}, K : (0, 1) \to (0, \infty)$ and every $\mu > 0$ there are constants $w_0 = w_0(\mu, \bar{r}, K) > 0$, $0 < s(\mu, \bar{r}, K) < \frac{1}{10}$ and $a(\mu) > 0$, monotone in μ, such that the following holds. Let (M, g) be a riemannian manifold and $T^2 \times [0, \infty) \approx M' \subset M$ be a closed subset such that*

(i) *$\exists A > 0$, $T^2 \times [A, \infty) \approx M'_A \subset M'$ is w_0-homothetic to a hyperbolic cusp of finite volume,*

(ii) *$\partial M'$ is convex, incompressible, of diameter $< w_0$, and has a neighbourhood in M diffeomorphic to $T^2 \times I$ which contains all points within distance less than 1 and $-\frac{5}{16} \leq \sec \leq -\frac{3}{16}$ there,*

(iii) *Each $x \in M'$ is w_0-thin,*

(iv) *For all $w \in (w_0, 1)$, $r < \bar{r}(w)$ and $x \in M'$ we have: if $\mathrm{vol}B(x, r) > wr^3$ and $r < \rho(x)$, then $|\mathrm{Rm}| \leqslant Kr^{-2}$, $|\nabla \mathrm{Rm}| \leqslant Kr^{-3}$ and $|\nabla^2 \mathrm{Rm}| \leqslant Kr^{-4}$ on $B(x, r)$.*

Then there are finitely many embedded 2-tori Σ_i^T and 2-spheres $\Sigma_i^S \subset M'$ which are pairwise disjoint and disjoint from $\partial M'$ as well as closed subsets $V_1, V_{1,cusp}, V_2, V'_2 \subset M'$ such that

(a_1) *$M' = V_1 \cup V_{1,cusp} \cup V_2 \cup V'_2$, the interiors of the sets $V_1, V_{1,cusp}, V_2$ and V'_2 are pairwise disjoint and $\partial V_1 \cup \partial V_{1,cusp} \cup \partial V_2 \cup \partial V'_2 = \partial M' \cup \bigcup_i \Sigma_i^T \cup \bigcup_i \Sigma_i^S$. No two components of the same set share a common boundary.*

(a_2) *$\partial V_1 \cup \partial V_{1,cusp} = \partial M' \cup \bigcup_i \Sigma_i^T \cup \bigcup_i \Sigma_i^S$. In particular, $V_2 \cap V'_2 = \emptyset$ and $V_2 \cup V'_2$ is disjoint from $\partial M'$.*

(a_3) *V_1 consists of components diffeomorphic to one of the following manifolds:*

$$T^2 \times I, S^2 \times I, D^2 \times I, D^3.$$

$V_{1,cusp}$ is connected and diffeomorphic to $T^2 \times [0, 1)$.

(a_4) *Every component of V'_2 has exactly one boundary component and this component borders V_1 on the other side. Moreover, every component of V'_2 is diffeomorphic to $D^2 \times S^1$ or D^3.*

[...] We also find closed subsets $V_{2,\mathrm{reg}}$, $V_{2,\mathrm{cone}}$, $V_{2,\delta} \subset V_2$ such that [...]

(b_2) *$V_{2,\mathrm{reg}}$ carries an S^1-fibration which is compatible with its boundary components and all its annular regions.*

[...]

(c_3) *For every $x \in V_{2,\mathrm{reg}}$ the ball $(B(x, s\rho_1(x)), s^{-1}\rho_1^{-1}(x), x)$ is μ-close to a standard 2-dimensional Euclidian ball $(B = B_1(0), g_{\mathrm{eucl}}, \bar{x} = 0)$. Moreover there is an open subset U with $B(x, \frac{1}{2}s\rho_1(x)) \subset U \subset B(x, s\rho_1(x))$ and a smooth map $p : U \to \mathbf{R}^2$ such that*

- (α.) *There are vector fields X_1, X_2 on U such that $dp(X_i) = \frac{\partial}{\partial x_i}$ and X_1, X_2 are almost orthogonal, i.e. $|\langle X_i, X_j\rangle - \delta ij| < \mu$ for $i = 1, 2$,*
- (β) *U is diffeomorphic to $B^2 \times S^1$ such that $p : U \to p(U)$ corresponds to the projection onto B^2 and the S^1-fibers are isotopic to the fibers of the fibration on $V_{2,\mathrm{reg}}$.*
- (γ) *The fibers of p as well as the fibers of $V_{2,\mathrm{reg}}$ on U have diameter at most μ and both families of fibers enclose an angle $< \mu$ with each other.*

[...] The rest of the proposition is similar to [Bam11a, Proposition 6.1], except that conclusion (c_1) applies to V_1 and $V_{1,cusp}$.

In our situation where $M' \approx T^2 \times [0, \infty)$, conclusions ($a_1$), ($a_2$) imply that one of the following holds:

- $M' = V_{1,cusp}$, or
- There is one component C_1 of V_1 with $\partial C_1 = \partial M'$, the cuspidal end of M' is covered by $V_{1,cusp}$ and there are components of V_2 bordering C_1 and $V_{1,cusp}$.

Indeed, in the second case a component which borders C_1 or $C_{1,cusp}$ cannot be in V_2', whose components have only one boundary component.

6 Proof of the main theorem

In this section we prove Theorem 1.2. We sketch the organisation of the proof. Let (M, g_0) be a riemannian 3-manifold satisfying the hypotheses of this theorem. The first step is to define a Ricci flow with $r(\cdot), \delta(\cdot)$-bubbling-off $g(\cdot)$, issued from g_0 and defined on $[0, +\infty)$. For this we may assume that M is nonspherical. We assume also that M does not have a metric

with $\mathrm{Rm} \geq 0$, otherwise it would be Seifert fibred. The existence on a maximal time interval $[0, T_{\max})$ is provided by the existence Theorem 2.15. As mentionned earlier, one may have to go to the universal cover. The case $T_{\max} < +\infty$ is easily ruled out using the fact that $(M, g(T_{\max}))$ is covered by canonical neighbourhoods (see claim 3 below). Then the thick-thin decomposition Theorem 3.3 can be used to construct decompositions $M = H_n \cup G_n$, where H_n is diffeomorphic to a complete, finite volume hyperbolic manifold H (maybe empty). Here we use results of Section 4 (precisely Proposition 4.1) to prove that tori of $\partial \overline{H}_n$ (if $H_n \neq \emptyset$) are incompressible in M. In this case, the atoroidality assumption on M implies that H_n is diffeomorphic to M and that each component of G_n is diffeomorphic to a cuspidal end $T^2 \times [0, \infty)$ of M_n. If this is true for a subsequence, then $g(t)$ converges (in the pointed topology, i.e. modulo pointed diffeomorphisms) to a complete, finite volume hyperbolic metric on M. In both cases ($H_n = \emptyset$ or $H_n \neq \emptyset$), G_n collapses with curvature locally controlled in the sense of Perelman. If $H_n = \emptyset$ for a subsequence, then we conclude by collapsing theorem 5.4 that $M_n = G_n$ is a graph manifold, hence Seifert fibred, for all n large enough. If $H_n \neq \emptyset$, Proposition 4.2 gives a continuous decomposition $M = H_t \cup G_t$ where H_t is diffeomorphic to M, $g(t)$ is smooth and $|\mathrm{Rm}| \leq Ct^{-1}$ there, and G_t is $\alpha(t)$-thin. Collapsing theory and Bamler's arguments are used to obtain that $t \mapsto g(t)$ is smooth and $|\mathrm{Rm}| \leq Ct^{-1}$ on G_t.

6.1 Setting up the proof

Let $(\tilde{M}, \tilde{g}_0)$ be the riemannian universal cover of (M, g_0). Observe that $\tilde{g}_0$ has bounded geometry. Without loss of generality, we assume that it is normalised. If M is compact, we can even assume that g_0 itself is normalised.

We now define a riemannian 3-manifold $(\bar{M}, \bar{g}_0)$ by setting $(\bar{M}, \bar{g}_0) := (M, g_0)$ if M is compact, and $(\bar{M}, \bar{g}_0) := (\tilde{M}, \tilde{g}_0)$ otherwise. In either case, $\bar{g}_0$ is complete and normalised. By [MSY82], $\bar{M}$ is irreducible. If $\bar{M}$ is spherical, then M is compact, hence $\tilde{M} = M = \bar{M}$ and the conclusion of Theorem 1.2 holds. Henceforth, we assume that $\bar{M}$ is nonspherical.

Thus Theorem 2.15 applies to $(\bar{M}, \bar{g}_0)$, where $\bar{\delta}(\cdot)$ is chosen from Theorem 3.3. Let $\bar{g}(\cdot)$ be a Ricci flow with bubbling-off on $\bar{M}$ with initial condition $\bar{g}_0$. By Addendum 2.17, we also have a Ricci flow with bubbling-off $g(\cdot)$ on M with initial condition g_0 covered by $\bar{g}(\cdot)$.

Claim 3. The evolving metrics $g(\cdot)$ and $\bar{g}(\cdot)$ are defined on $[0, +\infty)$.

Proof. If this is not true, then they are only defined up to some finite time T, and every point of $(\bar{M}, \bar{g}(T))$ is centre of an ε_0-neck or an ε_0-cap. By

Theorem 7.4 of [BBM11], $\bar{M}$ is diffeomorphic to S^3, $S^2 \times S^1$, $S^2 \times \mathbf{R}$ or $\mathbf{R}^3$.[1] Since $\bar{M}$ is irreducible and nonspherical, $\bar{M}$ is diffeomorphic to $\mathbf{R}^3$. The complement of the neck-like part (cf. again [DL09]) is a 3-ball, which must be invariant by the action of the deck transformation group. Since this group acts freely, it is trivial. Thus $M = \bar{M}$.

Being covered by $\bar{g}(\cdot)$, the evolving metric $g(\cdot)$ is complete and of bounded sectional curvature. Hence by Remark 2.3, $(M, g(T))$ has finite volume. By contrast, $(\bar{M}, \bar{g}(T))$ contains an infinite collection of pairwise disjoint ε_0-necks of controlled size, hence has infinite volume. This contradiction completes the proof of Claim 3. $\square$

It follows from the claim that $\bar{M}$ carries an equivariant Ricci flow with bubbling-off $\bar{g}(\cdot)$ defined on $[0, +\infty)$ with initial condition $\bar{g}_0$, such that for every $k \geq 0$, the restriction of $\bar{g}(\cdot)$ to $(k, k+1]$ is a Ricci flow with $(r_k, \delta_k, \kappa_k)$-bubbling-off. We denote by $g(\cdot)$ the quotient evolving metric on M. By Addendum 2.17, it is a Ricci flow with $r(\cdot), \delta(\cdot)$-bubbling-off. By Theorem 2.19, $g(\cdot)$ remains cusp-like at infinity for all time. Now consider the alternative that follows the conclusion of Theorem 3.3: Either

- there exist $w > 0$, $t_n \to \infty$ such that the w-thick part of $(M, t_n^{-1} g(t_n))$ is non empty for all n, or
- there exist $w_n \to 0$, $t_n \to \infty$ such that the w_n-thick part of $(M, t_n^{-1} g(t_n))$ is empty for all n.

We refer to the first case as the *noncollapsing case* and to the second as the *collapsing case*.

We denote by g_n the metric ${t_n}^{-1} g(t_n)$. Note that g_n has curvature locally controlled in the sense of Perelman (cf remark 5.2). We denote by M_n the riemannian manifold (M, g_n), $M_n^+(w)$ its w-thick part, and $M_n^-(w)$ its w-thin part. In the collapsing case, $M_n = M_n^-(w_n)$ fits the assumptions of Theorem 5.4 ([MT08] Theorem 0.2 in the compact case), hence is a graph manifold for n large enough.

Let us consider the other case.

6.2 The noncollapsing case

By assumption, there exist $w > 0$ and a sequence $t_n \to \infty$ such that the w-thick part of M_n is nonempty for all n. Choose a sequence $x_n \in M_n^+(w)$. By

[1] This list is shorter than the corresponding list in [BBM11] since we do not consider caps diffeomorphic to the punctured RP^3.

part (2) of the thick-thin decomposition Theorem 3.3, (M_n, x_n) subconverges to a complete hyperbolic manifold $(H, *)$ of finite volume. Suppose that H is not closed. By definition of the convergence, there exist, up to extracting a subsequence, an exhaustion of H by compact cores $\bar{H}_n \subset H$ and embeddings $f_n : (\bar{H}_n, *) \to (M, x_n)$ such that $|g_{\mathrm{hyp}} - f_n^* g_n|$ goes to zero. By Proposition 4.1, for each $m \in \mathbf{N}$, for all n large enough, each component of $f_n(\partial \bar{H}_m)$ is an incompressible torus in M. Relabeling the f_n we can assume that the property holds for $f_m(\partial \bar{H}_m)$ for all m. By atoroidality of M, it follows that $H_n := \mathrm{int}\, f_n(\bar{H}_n) \subset M$ is diffeomorphic to M for all n, and $G_n := M \setminus H_n$ is a disjoint union of neighbourhoods of cuspidal ends of M_n. Proposition 4.2 (stability of the thick part) gives $T_0 > 0$ and a nonincreasing function $\alpha : [T_0, \infty) \to (0, \infty)$ tending to zero at infinity, and for $t \geq T_0$ embeddings $f(t) : B(*, \alpha(t)^{-1}) \subset H \to M$ satisfying conclusions (i)–(iii) of the proposition. For each $t \geq T_0$, choose a compact core $\bar{H}_t \subset B(*, \alpha(t)^{-1})$ such that $\partial \bar{H}_t$ consists of horospherical tori whose diameter goes to zero as $t \to \infty$. We assume moreover that $t \to \bar{H}_t$ is smooth. Let $H_t = f(t)(\bar{H}_t)$ and $G_t = M \setminus H_t$. Then H_t is diffeomorphic to M, $t \mapsto g(t)$ is smooth there and $|\mathrm{Rm}| \leq Ct^{-1}$ by closeness with H. On the other hand G_t is $w(t)$-thin for some $w(t) \to 0$ as $t \to \infty$. There remains to prove that G_t is unscathed and satisfies $|\mathrm{Rm}| \leq Ct^{-1}$ also. The unscathedness will follow from the curvature estimate, as surgeries require large curvature.

In the case where H is closed, Proposition 4.2 gives $H_t = M$ and the conclusion follows. From now we assume that H is not closed.

Let $\bar{r}$, K be functions given by Theorem 3.3, $\mu_0 > 0$ given by [Bam11a, Lemma 7.1], let $w_0 = w_0(\bar{r}, K, \mu_0)$ be given by Proposition 5.7.

Consider a connected component $\mathcal{C}(t)$ of G_t. Section 4 proves that ∂G_t are incompressible tori in M for large t, hence $\mathcal{C}(t)$ is diffeomorphic to $T^2 \times [0, \infty)$. Moreover $\mathcal{C}(t)$ is $w(t)$-thin for some $w(t) \to 0$ hence satisfies assumptions of Proposition 5.7 for all t large enough (in particular $w(t) < w_0$ for $t \geq T_1 \geq T_0$). Then by Proposition 5.7 decompose $\mathcal{C}(t)$ in sets $V_1, V_{1,\mathrm{cusp}}, V_2, V_2'$ (which depend of t). As already observed, either $\mathcal{C}(t) = V_{1,\mathrm{cusp}}$ or there exist components $\mathcal{C}_1 \subset V_1$ and $\mathcal{C}_{1,\mathrm{cusp}} = V_{1,\mathrm{cusp}}$ such that $\partial \mathcal{C}_1 = \partial \mathcal{C}(t)$, $\mathcal{C}_{1,\mathrm{cusp}}$ covers the cuspidal end of $\mathcal{C}(t)$, and $\mathcal{C}_1$ and $\mathcal{C}_{1,\mathrm{cusp}}$ are bordered by components of V_2. We intend to prove that only the first possibility occurs, that is $\mathcal{C}(t) = V_{1,cusp}$ for all t large enough. This will follow by a contradiction argument using the areas of homotopies connecting a noncontractible loop in $\mathcal{C}_1$ to a noncontractible loop in $\mathcal{C}_{1,\mathrm{cusp}}$. The area estimate will prevent the homotopies to intersect the V_2 components bordering $\mathcal{C}_1$ or $\mathcal{C}_{1,\mathrm{cusp}}$.

Before doing this observe that [Bam11a, Lemma 7.1] applies to any $x \in V_{1,\mathrm{cusp}}$, giving

$$\mathrm{vol} \tilde{B}(\tilde{x}, t, \rho_{\sqrt{t}}(x, t)) \geq w_1(\rho_{\sqrt{t}}(x, t))^3.$$

Proposition 3.2 then implies $|\operatorname{Rm}| \leq K(w_1)t^{-1}$ there for t large enough, say $t \geq T_2 = \max(T_1, T(w_1)$. Hence we can conclude the proof of the theorem if $\mathcal{C}(t) = V_{1,\text{cusp}}$ for all t large enough.

Assume by contradiction that $\mathcal{C}(t) \neq V_{1,\text{cusp}}$ for some t, and let $\mathcal{C}_2 \subset V_2$ be a connected component which borders $\mathcal{C}_1$ or $\mathcal{C}_{1,\text{cusp}}$, say $\mathcal{C}_1$. Observe that its generic S^1-fibres are homotopic to a nontrivial curve in $\partial\mathcal{C}_1$, by 5.7 (b_2). By incompressibility of $\partial\mathcal{C}_1$ in M, the curve generate an infinite cyclic subgroup in $\pi_1 M$. Then [Bam11a, Lemma 7.1] and Proposition 3.2 apply also to $\mathcal{C}_2$ and give $\rho_{\sqrt{t}} \geq \bar{\rho}\sqrt{t}$ there. Moreover for $x \in \mathcal{C}_2 \cap V_{2,\text{reg}}$, Proposition 5.7 (c3) gives an open set U such that

$$B(x, t, \frac{1}{2}s\rho_{\sqrt{t}}(x,t)) \subset U \subset B(x, t, \rho_{\sqrt{t}}(x,t))$$

and a 2-Lipschitz map $p : U \to \mathbf{R}^2$ whose image contain $B(0, \frac{1}{4}s\rho_{\sqrt{t}}(x,t)) \subset \mathbf{R}^2$ and whose fibres are homotopic to fibres of $\mathcal{C}_2$, hence noncontractible in M.

Consider a noncontractible loop $\gamma \subset \mathcal{C}(T_2)$. Define for $t \geq T_2$, $\gamma_1(t) \subset \partial\mathcal{C}(t)$ freely homotopic to γ and such that $f(t)^{-1} \circ \gamma_1(t)$ is geodesic in $\partial\bar{H}_t$ and evolves by parallel transport in H w.r.t. t. On the side of the cusp, using the cusp-like structure assumption and theorem 2.19 there is an embedding $f_{\text{cusp}} : T^2 \times [0, +\infty) \to M$ and a function $b : [0, +\infty) \to [0, \infty)$ such that

$$|t^{-1} f_{\text{cusp}}^* g(t) - g_{\text{hyp}}|_{T^2 \times [b(t), +\infty)} < w(t)$$

where g_{hyp} denotes the hyperbolic metric $e^{-s} g_{\text{eucl}} + ds^2$ on $T^2 \times [0, +\infty)$. A priori this metric may differ from the one on H. Then define $\mathcal{C}_{\text{cusp}}(t) = f_{\text{cusp}}(T^2 \times [b(t), +\infty))$. We can assume $b(t) \to \infty$ as $t \to \infty$ so that $\mathcal{C}_{\text{cusp}}(t) \subset \mathcal{C}_{1,cusp}$ when this set is defined. Then define $\gamma_2(t) \subset \partial\mathcal{C}_{\text{cusp}}(t)$ freely homotopic to γ such that $f_{\text{cusp}}^{-1} \circ \gamma_2(t) \subset T^2 \times \{b(t)\}$ is geodesic in (T^2, g_{eucl}) and evolves by parallel transport (at speed b').

In particular $\gamma_1(t) \subset \mathcal{C}_1$ and $\gamma_2(t) \subset \mathcal{C}_{1,\text{cusp}}$ when these sets are defined, and these loops are freely homotopic. Let $A(t)$ be the infimum of the areas of all smooth homotopies $H : S^1 \times [0,1] \to \mathcal{C}(t)$ connecting $\gamma_1(t)$ to $\gamma_2(t)$.

Claim 4. $t^{-1}A(t) \to 0$ as $t \to \infty$.

Let us conclude the proof of the theorem before showing the claim. Consider γ, β smooth loops in $\mathcal{C}(t)$ generating $\pi_1\mathcal{C}(t)$. Let $\gamma_i(t)$, resp. $\beta_i(t)$, $i = 1, 2$, defined as above freely homotopic to γ, resp. β. Let $A(t)$, resp. $B(t)$, be the infimum of the areas of all smooth homotopies connecting $\gamma_1(t)$ to $\gamma_2(t)$, resp. $\beta_1(t)$ to $\beta_2(t)$. By the above claim

$$t^{-1}A(t) + t^{-1}B(t) \to 0 \tag{7}$$

as $t \to \infty$. On the other hand let H_γ, resp. H_β, be any of these homotopies. Recall that any fibres of the projection $p : U \to \mathbf{R}^2$ is a noncontractible loop $\subset \mathcal{C}_2$, hence it intersects at least one the homotopies H_γ,H_β. It follows, using the fact that p is 2-bilipschitz, that

$$\text{area}(H_\gamma) + \text{area}(H_\beta) \geq \frac{1}{4}\text{vol}(p(U)) \geq cs^2\bar{\rho}t,$$

for all t, contradicting (7)

It remains to prove the claim.

Proof of the claim. It is identical to [Bam11a, Lemma 8.2], except that we have to account for the fact that $\partial_t\gamma_2(t)$ may not be a priori bounded. This estimate appears when we compute the area added to the homotopy by moving the boundary curves. The infinitesimal added area to the homotopy due to the deplacement of γ_1 is negative (we can assume $\alpha' > 0$), hence neglected. The contribution of γ_2, by closeness with the hyperbolic cusp, is bounded by $Ct.e^{-b}b'$. On the other hand, the normalised length $t^{-1/2}\ell(\gamma_i) \to 0$ and the normalised geodesic curvature $t\kappa(\gamma_i(t)) < C$, by closeness with the hyperbolic situation. Let us denote $L(t) = t^{-1/2}(\ell(\gamma_1(t)) + \ell(\gamma_2(t))$. Computations in [Bam11a, Lemma 8.2] give (compare with (8.1))

$$\frac{d}{dt^+}(t^{-1}A(t)) \leq -\frac{A(t)}{4t^2} + C\left(\frac{L(t)}{t} + e^{-b}b'\right) \tag{8}$$

Denoting $y(t) = t^{-1}A(t)$ this gives the differentiel inequation

$$y' \leq -y/4t + C(t^{-1}L + e^{-b}b')$$

Using the variation of the constant method, one obtains that $y(t) = K(t)t^{-1/4}$ where $K' \leq Ct^{1/4}\left(t^{-1}L + e^{-\alpha}\alpha' + e^{-b}b'\right)$. By definition of $\gamma_i(t)$, for $T_2 \leq a \leq t$, $\int_a^t L(t)\ dt \leq C\int_a^t e^{-\alpha}\alpha' + e^{-b}b'\ dt \leq C(e^{-\alpha(a)} + e^{-b(a)}) \to 0$ as $a \to \infty$. Then for $t \geq a$ one has

$$K(t) - K(a) \leq C\left(a^{-3/4}\int_a^t L(t)\ dt + t^{1/4}e^{-b(a)}\right)$$

hence

$$\begin{aligned} y(t) &= \frac{K(a)}{t^{1/4}} + \frac{K(t) - K(a)}{t^{1/4}} & (9) \\ &\leq \frac{K(a)}{t^{1/4}} + C\left(\frac{a^{-3/4}\int_a^t L(t)\ dt}{t^{1/4}} + e^{-b(a)}\right) & (10) \end{aligned}$$

which is arbitrary small by taking a then t large enough. ∎

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